INTRODUCING THE ANTHROPOLOGICAL THEORY OF THE DIDACTIC: AN ATTEMPT AT A PRINCIPLED APPROACH

Yves Chevallard
Professor emeritus at Aix-Marseille University
Marseille, France

This paper brings together the substance of a series of interventions and discussion sessions focusing on the principles and use of the anthropological theory of the didactic that took place in Osaka (Japan) between October 8 and October 12, 2016. YC.

1. SOME INTRODUCTORY REMARKS

Dear colleagues, ladies and gentlemen, my first words will be to thank the distinguished Japanese colleagues who have taken it upon themselves to invite me to deliver a lecture today here in Osaka[1]. I also want to thank you all for being here today. And I wish to apologize for not being able to deliver my presentation in your mother tongue. By the same token let me apologize in advance to our wonderful translators for the puzzlement the involuntary intricacies of my talk may occasion.

My aim in this lecture will be to paint an overall picture of the present state of the so-called *anthropological theory of the didactic*, or ATD for short. In trying to do so, I shall focus mainly on those points on which ATD departs most from the more widespread, ingrained approaches to education in general and to mathematics teaching in particular.

Before I start on this short, casual journey, two or three remarks are in order. Firstly, although ATD has emerged as a theory of *mathematics* education, its construction over the last four decades has been aimed at providing us with adequate means of explaining and understanding the teaching and learning of *any* kind of knowledge, considered generally *and* specifically, be it mathematics, English, biology, history, or what have you.

My second remark will be about something a little unsatisfying: in many cases—not all!—, I shall have to skip details and examples in order to highlight what I believe to be the essential aspects of the ATD approach.

One more point before I really start lecturing. Every time I prepare (for) a lecture in English, which is neither my mother tongue nor my usual working language, as you can surmise from these introductory remarks, I have a quick look at a number of papers purporting to advise the unpretentious lecturer about the risks of the trade. Referring to the always lurking risk of cultural misunderstanding, one of them ("Presentation

1 I particularly thank my Japanese colleagues Tatsuya Mizoguchi, Yusuke Shinno, Takeshi Miyakawa, Koji Otaki, and Yoshitaka Abe. Last but not least, I warmly thank my longtime friend, colleague, and scientific partner Marianna Bosch, without whose unrelenting support and incisive mind little would have been achieved.

delivery," n.d.) alludes to the case of "a German working for an Israeli company making a presentation in English to a Japanese audience in Korea." Ours is a formally simpler situation, of course! But anyway, while trying to avoid unsuspected causes of misunderstanding, I want to apologize for all cases of "cultural incorrectness" that might occur. Cultural adequacy, I am aware, is not so easy to achieve. According to the same online paper, "to a Latin from Southern France or Italy, a presenter who uses his hands and arms when speaking may seem dynamic and friendly" whereas "to an Englishman, the same presenter may seem unsure of his words and lacking in self-confidence." Well, we shall see.

2. THE STARTING POINT: DIDACTIC TRANSPOSITION THEORY

The first buds of the anthropological theory of the didactic are to be found in what is known under the name of *didactic transposition theory*, which is now a *subtheory* of ATD. The model on which this theory rests sounds fairly simple. First and foremost, let us consider some developed human society $\check{S}$, for example the society of a nation-state, such as the Japanese society, the Spanish society, or the French society[2]. Within such a society, we suppose that there exists a number of institutions called *school systems*, each of them with a number of (physically) different *school establishments* (which we will simply call "schools" for short). There is, for example, the primary and the secondary school system for general education. As a rule, for every occupational pursuit, there is at least one special training school system. As is usual with ATD, the expression "school system" has a very broad meaning: it embraces a wide variety of institutions, such as the family, sports associations, bars or street gangs. De jure or de facto, all these social entities and structures assume a formative role and thereby serve willy-nilly a scholastic function.

Let us denote such a school system by the Greek capital letter Σ (sigma), if only because Σ is the initial letter of the Greek word *scholē* (σχολή), from which the word *school* derives. Likewise, I denote a school establishment belonging to the school system Σ by the Greek small letter σ (sigma), and I write: $\sigma \in \Sigma$. According to the *Online Etymology Dictionary* ("School," 2001-2016), the original meaning of the word *scholē* in ancient Greek was "'leisure,' which passed to 'otiose discussion' (in Athens or Rome the favorite or proper use for free time), then 'place for such discussion.'" The Greek word was rendered into Latin by *schola*, a word inherited by most modern European languages, for example English (*school*), French (*école*) or Spanish (*escuela*). Since ancient Greece, a school is therefore a "place for instruction," in which mundane occupations are temporarily disregarded and where studying prevails over worldly interests.

A school σ is the place where *didactic systems* live (and die). A didactic system denoted by $S(X, Y, \mathscr{k})$ is made up of a set—or class[3]—of students X, a team of teachers and other school helpers Y, and some piece of knowledge $\mathscr{k}$ taken from a *body of knowledge* $\mathcal{K}$ or *discipline* $\mathcal{D}$: $\mathscr{k} \subseteq \mathcal{K} \subseteq \mathcal{D}$. The *discipline* $\mathcal{D}$ may be, for instance, Mathematics, Physics, English, History, Photography, etc. In a given society $\check{S}$, the (fuzzy) set of "academic" disciplines $\mathcal{D}$, that is of bodies of knowledge $\mathcal{K}$ regarded as worthy of being taught in some school systems, does depend on $\check{S}$. For the society $\check{S}$ may consider a field of human activity (say, soccer, or

2 The symbol $\check{S}$ is read "s (with) caron" or "s (with) hatchek".

3 The word "class" is a polysemic word. In this paper, it will be applied to X as well as to the ordered pair denoted by $[X, Y]$.

sumo, or opera singing) as relying on a "true" body of knowledge—*or not*. Of course, ATD uses a very wide notion of knowledge in order to take account of the full variety of different societies' explicit (or implicit) stances on the nature of knowledge.

Let us now consider a didactic system $S(X, Y, \mathcal{k})$ in a school σ within a school system Σ, in a given society $\check{S}$. One crucial question is: Who exactly are the persons x that can be members of the class X? (Beware, in ATD a *person* may be a newborn baby, no less than a full-fledged adult.) For sure, not every person can appear as a student x in $S(X, Y, \mathcal{k})$: as we all know, there is a whole gamut of conditions—of age, sex, diploma, etc.—that a person must satisfy to become an $x \in X$. The same holds for becoming a $y \in Y$. Now the theory of didactic transposition raises yet another key question, the question of the precise nature of the piece of knowledge $\mathcal{k}$ which is the "didactic stake"—the thing to be taught and learned—in $S(X, Y, \mathcal{k})$. Before we go any further in this respect, I want to stress that raising this question is very unlikely, because the "true" nature of $\mathcal{k}$ is usually *taken for granted* and remains unquestioned. Let me denote by z any person involved in the teaching of mathematics in $\check{S}$. If, for example, such a person z is reminded that, in grade 9, students have to cope with quadratic equations, we are sure that z will *not* wonder about "what exactly" a quadratic equation is. Even if z has never heard of it, the mainstream behavior in this respect consists in assuming that a quadratic equation is something well-defined, self-evident, and well-known to most people!

At the heart of didactic transposition theory lies this far-fetched, unassuming, but indeed quite far-reaching question: *What is this thing you call* $\mathcal{k}$*?* Normally any z to whom the question is put will exclaim: "But you know what it is, don't you? You are a math teacher, aren't you? Any expert on $\mathcal{D}$ knows what $\mathcal{k}$ is." The ATD-didactician's rejoinder will then be: "Well, I do know what the label $\mathcal{k}$ means in *my* world, which is for sure a subworld of $\check{S}$. But I don't know what it refers to in the didactic system $S(X, Y, \mathcal{k})$ of the school σ within the school system Σ." This uneasy dialogue may go on like this—where $\hat{z}$ denotes the ATD-didactician:

> "The piece of knowledge $\mathcal{k}$ belongs to the discipline $\mathcal{D}$, and I know that you know $\mathcal{D}$, for they told me you are a $\mathcal{D}$-ist," z retorted.
> "For sure I know a scientific discipline labeled $\mathcal{D}$. But is it the same discipline $\mathcal{D}$ that the people in σ have in mind?" $\hat{z}$ observed.
> "I suppose so," z grumbled.

This debate resembles a dialogue of the deaf in which $\hat{z}$ occupies a minority position—in fact, for most people, "$\mathcal{k}$ is $\mathcal{k}$", period.

In order to understand what really occurs, we have to distinguish—in contrast to the majority belief—between different "versions" of $\mathcal{k}$. On the one hand, we must acknowledge the version which "lives" in the school σ, which we shall denote by $\mathcal{k}_\sigma$. On the other hand, we have to consider, with the ATD-didactician $\hat{z}$, another version of $\mathcal{k}$ which we call a *scholarly* version, denoted metaphorically by $\mathcal{k}_\infty$, where ∞ is the symbol of infinity[4]. At this point, the theory of didactic transposition posits two key tenets. The first one is that $\mathcal{k}_\sigma$ and $\mathcal{k}_\infty$ are almost always different: $\mathcal{k}_\sigma \neq \mathcal{k}_\infty$. The second principle is that this difference between $\mathcal{k}_\sigma$

4 The symbol $\hat{z}$ is to be read "z circumflex" or (in mathematics) "z hat."

and $\mathscr{k}_\infty$ is denied by almost all the people concerned by $\mathscr{k}_\sigma$, especially the teachers of $\mathcal{D}$ who have to teach $\mathscr{k}$ (or, more exactly, some version $\mathscr{k}_\sigma$ of $\mathscr{k}$). The theory of didactic transposition has an explanation for both phenomena.

The first point should be first generalized a bit: in every institution I where the piece of knowledge $\mathscr{k}$ "lives," it exists in a version specific to I, which we can generically denote by $\mathscr{k}_I$. As a general rule, in an institution I, one of three things can happen: the piece of knowledge $\mathscr{k}$ can be either *created* in I (which is therefore a scholarly institution I_∞ and the birthplace—or one of the birthplaces—of $\mathscr{k}$), or *used* ($\mathscr{k}$ is a tool in I), or *taught* (as is the case in σ). Now the *conditions* under which $\mathscr{k}$ lives in I vary according to I. For example, $\mathscr{k}_\infty$ was created in I_∞ by adults who were experts in $\mathcal{D}$—they are, for example, mathematicians by trade—while $\mathscr{k}_\sigma$ is to be taught, say, to ninth graders, who are usually 14-15 years old, which makes a difference. More generally, in every institution I which "imports" $\mathscr{k}$ from some other institution, say I_∞, $\mathscr{k}$ will have to adapt to the conditions prevailing in I, and therefore to the epistemological *ecology* of I. Classically this process should "copy" $\mathscr{k} = \mathscr{k}_\infty$ in I_∞ to "paste" it into σ. But the passage from $\mathscr{k}_\infty$ to $\mathscr{k}_\sigma$ is not a true copy and paste process: it entails a number of distortions that make $\mathscr{k}$ into a near-copy viable in σ. We summarize it by saying that the process through which $\mathscr{k}_\infty$ is changed into $\mathscr{k}_\sigma$ is a *transposition process*— $\mathscr{k}_\infty$ is said[5] to be "transposed" into $\mathscr{k}_\sigma$. Because, in this case, σ happens to be a didactic institution, the transposition process is labeled a *didactic* transposition process. When the transposition takes place between an institution I and an institution J which is not regarded as "didactic," we simply speak of an *institutional* transposition process, or transposition process for short.

Let me dwell for a moment on the "distortions" that $\mathscr{k}$ may undergo. One aspect is easily understandable: it is the quantitative aspect. Suppose mathematicians in I_∞ have created an axiomatic theory $\mathscr{k}_\infty$ of something—for example, of Euclidean geometry—which comprises some 20 axioms—as in the case of David Hilbert's axiomatization of geometry. If, in the school system Σ (or in any of its school establishments σ), such a list of axioms is considered "unteachable", because, due to its length and complexity, it is not regarded as learnable by the students in the didactic systems of σ, then either $\mathscr{k}_\infty$ will never be transposed into σ or the transposition process will bring about significant changes, which in the worst cases will denature $\mathscr{k}_\infty$, thus making $\mathscr{k}_\sigma$ unfit for the uses originally made of $\mathscr{k}_\infty$ in I_∞—for example by depleting a "scholarly" axiomatic of Euclidean geometry to such an extent that the most common—and self-obvious—properties of Euclidean space are no longer derivable from the newly transposed set of axioms, unless you give up on mathematical rigor.

Problems of length and complexity (or subtlety) are main factors and incentives of didactic transposition. I will mention here a small set of easy examples, linked to the question of square roots. It seems that, for reasons deep-rooted in the curricular tradition, but maybe not for deep reasons, the traditional mathematics curriculum $\mathcal{C}$ treats square roots ambiguously. The theme of square roots appears, indeed, to be a "frontier," that is to say—to use metaphors from topology—a place near the boundary between the interior and the exterior of $\mathcal{C}$ where life is hard and uncertain because one false step can lead you astray into unchartered territory. Let us consider here the case of algebraic equations. We know that equations of degree one, two,

5 The online *Macmillan Dictionary* defines "to transpose" as meaning "to use an idea or method in a different situation from the one it was originally developed in."

three, or four can be solved algebraically, whereas this is not true in general of equations of degree five and above. Equations of degree one with rational coefficients belong, beyond doubt, to the interior of $\mathcal{C}$. Similarly, equations of degree three or four belongs to its exterior, The reason to ignore these memorable pieces of algebraic knowledge has been known for ages, if I may say so. For example, in *An Elementary Treatise on Algebra* published in Cincinnati (Ohio) in 1845 by Ormsby MacKnight Mitchel (1810-1862), “professor of mathematics and natural philosophy” at Cincinnati College, one reads the following:

> Thus far we have only examined questions giving rise to either simple or quadratic equations. These two classes of equations admit of direct modes of solution, more simple and easily applied than any indirect methods. This is not true of equations of a higher degree than the second. Direct methods have been diiscovered [sic] for solving equations of the third and fourth degrees; but they are far more complex and intricate than the indirect methods, which possess, likewise, the advantage of being equally applicable to equations of any degree. (p. 181)

However, it is no true that *all* kinds of quadratic equations belong to the interior of $\mathcal{C}$. Certainly, equations like $2x^2 + x = 0$, which boils down to first degree equations, are center-stage in $\mathcal{C}$. The same occurs with an equation like $2x^2 + x - 1 = 0$, whose roots are $x = -1$ and $x = \frac{1}{2}$. But let us consider now the equation $2x^2 + x - 2 = 0$. This equation, whose roots are $x = -\frac{1+\sqrt{17}}{4}$ and $x = \frac{\sqrt{17}-1}{4}$, is precisely on the “boundary” of $\mathcal{C}$. Now a slight change in the constant coefficient will give rise to an equation which stands clearly in the exterior of $\mathcal{C}$. If we consider the equation $2x^2 + x - \sqrt{2} = 0$, we get roots, to wit $x = -\frac{\sqrt{8\sqrt{2}+1}+1}{4}$ and $x = \frac{\sqrt{8\sqrt{2}+1}-1}{4}$, that are far too complicated to be the end-result expected even from the average twelfth-grader. If, by contrast, we consider the quadratic equation $2x^2 + \sqrt{2}\,x - \frac{1}{4} = 0$, we get back into the boundary of $\mathcal{C}$: this time, the roots are $x = -\frac{2+\sqrt{2}}{4}$ and $x = \frac{2-\sqrt{2}}{4}$. (The eagle-eyed reader will have noticed that the equation $2x^2 + x - \sqrt{2} = 0$ can be rewritten in the following manner: $X^2 + X - \frac{1}{4} = 0$, with $X = \sqrt{2}\,x$, a fact that explains much of the observed results.)

It is very noteworthy that, as is the case with quadratic equations, when observing some $\mathscr{k}_\sigma$ punctiliously, we almost always stumble upon an unexpected piece of institutional bric-a-brac, a true epistemological curio. Let me add here another example regarding square roots. At an earlier age—maybe in grade 8—, students have to grapple with trying to factor algebraic expressions. Let us consider the expression $x^2 - 4$. Students learn to look at it as a “difference of squares” and, consequently, will write: $x^2 - 4 = (x - 2)(x + 2)$. But generally they are told *not* to look at the expressions $x^2 - 3$ or $x^2 - 5$ as differences of squares, even after they have studied square roots. Students must understand that a difference of squares is a difference of *perfect* squares (“Rules of Factoring,” n.d.): while they can solve the equation $x^2 - 4 = 0$, they are denied the right to solve the equations $x^2 - 3 = 0$ or $x^2 - 5 = 0$. In the same way, while they are expected to factor the expression $4x^2 - 9$, thereby solving the equation $4x^2 - 9 = 0$, they are definitely not allowed to consider the equality $5x^2 - 14 = (\sqrt{5}\,x - \sqrt{14})(\sqrt{5}\,x + \sqrt{14})$, which shows that the roots of the equation $5x^2 - 14 = 0$ are

$x = \pm\sqrt{2{,}8} \approx \pm\, 1{,}67332$.

The second principle I have mentioned says that an overwhelming majority of people take for granted—to the point that it is rarely if ever put in question—that $\Bbbk_\sigma$ is "essentially" the same as $\Bbbk_\infty$; or, if I may venture a mathematical metaphor, that $\Bbbk_\sigma$ is asymptotically equal to $\Bbbk_\infty$: $\Bbbk_\sigma \simeq \Bbbk_\infty$. Why is it so? Whenever a school system Σ wishes to teach some piece of knowledge $\Bbbk$ in its schools σ, Σ has to be given the permission to do so by a number of relevant authorities in $\check{S}$—the Ministry of Education for example. One fundamental aspect of such an authorization is the requirement that $\Bbbk_\sigma$ be granted *epistemological legitimacy*, which is a twofold requisite. On the one hand, $\Bbbk_\sigma$ must be regarded as "sound knowledge"—not "fake" knowledge, but "real" knowledge—, which generally amounts to being accredited by the discipline $\mathcal{D}$ seen as an epistemological guarantor. On the other hand, $\Bbbk_\sigma$ must be faithful to what it is purported to be. For example, if Σ claims to teach a chunk of Euclidean geometry $\Bbbk_\sigma$, it is required a) that $\Bbbk_\sigma$ qualify as being true mathematical knowledge, and b) that this mathematical knowledge be recognized as being Euclidean geometry. In other cases, Σ's claim to teach Euclidean geometry would sound like a "labeling fraud" and this would challenge the *didactic* legitimacy of Σ, undermining its future as a teaching institution.

By contrast, the two requirements imposed upon $\Bbbk_\sigma$ can be fulfilled if $\Bbbk_\sigma$ is validated and accredited by the supreme authority in the field—the *scholarly* institutions I_∞ creatively in charge of the discipline $\mathcal{D}$. A caveat is in order at this point. The word "scholar" is here the rendering of the French word *savant*, which means "a knowledgeable person, one who has a lot of knowledge." In the case of mathematics, today, the "scholars" (or "savants") are without a doubt the professional mathematicians, who, in universities and other mathematics research centers, create new mathematical knowledge and, directly or indirectly, warrant—or sometimes denounce—the "mathematical genuineness" of taught knowledge. As it is most often the case with ATD, the word "scholar" should be taken in the broadest possible sense: contrary to its use where mathematics, or physics, or history are concerned, where it often sounds like a value-loaded notion, "scholar" should be construed as value-free. The main postulate of ATD in this respect is that *any* human activity, be it, say, carpentry or welding, not only mathematics or science, has its own scholars, held to "know best" than the rest of the people in the little world where they belong—they are big people in their little world. In actual fact, this applies, it seems, to all little worlds, including the world of mathematics and mathematicians, if we consent to ignore the egotism and parochiality that take their toll on most human groups.

Let me conclude this part of my talk. In order to become teachable (and learnable) in some school system Σ, a piece of knowledge $\Bbbk$ has to be "transposed" from some hypothetical scholarly world to adapt to conditions specific to the schools $\sigma \in \Sigma$. This "distortion" process, which often concocts unprecedented and unanticipated pieces of knowledge $\Bbbk_\sigma$, cannot be circumvented. Its end-result $\Bbbk_\sigma$ will be regarded as "acceptably true" to $\Bbbk_\infty$ if it is found to be a "fair copy" of $\Bbbk_\infty$ or, even better, if it is recognized as "asymptotically equal" to $\Bbbk_\infty$. In this respect, the first questions that ATD leads us to raise are: *What is this piece of knowledge that you call $\Bbbk_\sigma$ and claim to teach? Where does $\Bbbk_\sigma$ come from? How is $\Bbbk_\sigma$ legitimized—epistemologically speaking? Is $\Bbbk_\sigma$ viable in the long run? Or will it have to be reprocessed or even dislodged?*

Using another metaphor from mathematics, we may summarize the preceding discussion by saying that, in order to be epistemologically viable, $\Bbbk_\sigma$ must be such that the distance denoted by $d(\Bbbk_\sigma, \Bbbk_\infty)$ and supposed to measure the overall similarity between $\Bbbk_\sigma$ and $\Bbbk_\infty$ is *small enough*, so that scholarly institutions can in good conscience shield taught knowledge from epistemology-driven criticisms. However $\Bbbk_\sigma$ is jeopardized by a threat not yet mentioned. The distance $d(\mathscr{k}_\sigma, \mathscr{k}_\mu)$ between $\mathscr{k}_\sigma$ and the "corresponding" piece of knowledge $\mathscr{k}_\mu$ in the milieu μ where the students concerned live outside σ *must not be too small*, lest people in μ find the teaching of $\mathscr{k}_\sigma$ didactically pointless because $\mathscr{k}_\sigma$ is considered well-known in μ—to them it's nothing new under the sun! If, however, the teaching of $\mathscr{k}_\sigma$ is not disapproved, the school σ will become the target of rampant criticisms raised by some people in μ against $\mathscr{k}_\sigma$ in the name of their self-declared expertise on precisely that piece of knowledge. This takes place for example when the school system Σ is the elementary-secondary school system, with μ the family (or even an extra-familial) environment. It will also be observed when Σ is some vocational school system, the knowledge imparted $\mathscr{k}_\sigma$ being commented upon by people in the trade, in the name of their working experience.

The didactic transposition process must therefore work miracles: its construct $\mathscr{k}_\sigma$ must at the same time look not too close to "common knowledge" (to escape generally unconstructive criticism from the surrounding milieus) and sufficiently close to the "scholarly" (in the broad sense clarified above) to be protected from less competent (although sometimes arrogant) attacks from laypeople who "speak their minds."

3. PERSONAL AND INSTITUTIONAL RELATIONS

Up to this point we have considered almost exclusively the notions of knowledge, piece of knowledge, and body of knowledge or discipline $\mathcal{D}$, taken to be primitive notions of the theory under consideration. Two other "ingredients" of the universe of didactic transposition have appeared repeatedly: *persons* (for example *students x* and *teachers y*) and *institutions I* (such as a *school system* Σ, a *school establishment* σ, or a social *milieu* μ). One more kind of hitherto unnamed entities has surreptitiously emerged: the different *positions p* that a person *x* may occupy in an institution *I*—the position of student in a class, of eldest son in a family, of boss in a company, etc. In what follows, we shall expand the theory so that the notion of knowledge will no longer remain a primitive, undefined notion.

While *knowledge* takes center stage in the theory of didactic transposition, what has been left implicit until now is *knowing*. In ATD, the answer to the question "What is knowing?" comes in two major, successive steps. In this section, we shall outline the *theory of personal and institutional relations*. In the next section, we shall introduce the *theory of praxeologies*—which, in the actual genesis of ATD, developed from the theory of personal and institutional relations.

To go further on this path, one must add to the notions of *person*, *institution*, and *position* the very general notion of an *object*. All four notions—person, institution, position, and object—turn out to be *interdefined*. Given an object *o* and a person *x*, we posit that there exists an entity—regarded, in what follows, as a set—called the *personal relation* of *x* to *o*. This set is denoted by $R(x, o)$ and comprises all the "ways" in which *x* relates to *o*—through pondering over *o*, speaking or writing about *o*, using *o*, handling *o*, dreaming or daydreaming or fantasizing about *o*, spiffing it up, etc. In an expanded sense of the verb *to know*, the relation of *x* to *o*, $R(x, o)$, encapsulates all that *x* "knows" about *o*. We shall say that *x knows o* or, dually, that *o exists* for *x* if $R(x, o) \neq \varnothing$. The reason for this minimalistic definition of *knowing* will soon become obvious. But, without further ado, it becomes possible to specify the notion of an object: in a society $\check{S}$, an object is any entity, whether material or immaterial, that exists for at least one person—or, equivalently, that is "known" to at least one person. Note that, according to this "definition", mathematics, geometry, and algebra are objects just as well as the percent (%), per mille (‰), and per ten thousand (‱) signs. As other notions of ATD, the notion of an object does not depend on the "size" of the object: it is, so to speak, "invariant by homothecy."

How does the personal relation $R(x, o)$ come into being? To give an answer to this question, we have to extend the notion of relation to an object to institutions I or, more exactly, to institutional positions p. We denote by $R_I(p, o)$ the *institutional* (or *positional*) relation of p to o. $R_I(p, o)$ describes what the personal relation of a person x to o should ideally look like if x were to occupy the position p in I. In such a case, we say that $R(x, o)$ conforms to $R_I(p, o)$ and write: $R(x, o) \cong R_I(p, o)$. When a person x occupies the position p in I, we say that x is *subjected* to p or, more concretely, to the institutional relations $R_I(p, o)$, for all the objects o that are known to p (i.e., such that $R_I(p, o) \neq \varnothing$): x is said to be a *subject* of I in the position p. If $R(x, o) \cong R_I(p, o)$, the person x is said to be a *good subject* of I in the position p with respect to o. Of course x will be deemed a *bad* subject if $R(x, o) \ncong R_I(p, o)$.

From cradle to grave, any person x submits willy-nilly to a host of institutional subjections—such as a newborn infant subjected to its mother's chirp and, before long, to the family language, or a child subjected to the position of pupil in a class, or a youth member of a neighborhood gang, etc. Given an individual x, it is the positional subjections to which x has been submitted so far that make x into a person. At each and every moment of their biography, persons are, so to speak, the "resultant" of the complex of institutional subjections to which they have been submitted. Their personal relations $R(x, o)$ arise from the various institutional relations $R_I(p, o)$ to which they are or have been subjected. Two points should be noted. On the one hand, any personal relation to an object is almost always heterogeneous, being the unplanned effect of many institutional subjections—a personal relation is almost always an accidental hybrid. On the other hand, a positional relation $R_I(p, o)$ is correlatively nobody's *personal* relation: in other words, the equality $R(x, o) = R_I(p, o)$ *never occurs*.

Two consequences are worthy of note. Firstly, in many institutions, a person x in the position p will be looked at by other persons (in the institution or outside it) as a "pure" subject in the position p in I, which reduces the person x to one particular subjection—a limiting factor when it comes to understanding and anticipating that person's behavior. In this respect, you can think of the way some mathematics teachers look at pupils—as if they were pure subjects of the math class, ignoring their concomitant subjections to the physics class, the English class, the history class, etc., and even more so, ignoring their outside subjections. As a general rule, when a person z observes, from a position q, a person x in the position p, z can only "see" a *part* of $R(x, o)$, which is called the *public part* of $R(x, o)$ with respect to the pair (p, q) of institutional positions.

A second consequence that I want to stress is the dialectical character of the interaction between persons and institutions. On the one hand, institutional relations shape and reshape personal relations. But, conversely, in the long run, the personal relations of the institution's subjects force its institutional relations to evolve—although this is not the only causal factor responsible for institutional change. Persons and institutions fashion one another in the course of time.

In all this, it is vital to keep in mind that, in contradistinction to the usual meaning of the word, a "subjection" should not be construed as something negative, but, much to the contrary, as an indispensable means for persons to gain power over the world around them—we become human beings by being subjected to the human world, in contrast to (supposedly) feral children[6], who are said to "live" in a world deprived of most humanizing subjections. At the same time, somewhat paradoxically, we get a feeling of freedom when we submit to a *new* subjection which will loosen former subjections that had become painful and burdensome. To quote only two examples, we can consider the case of a young person who, on falling in love, gets free—up to a point—from family attachments; or that of a researcher who, for a long time subjected to a dominant theory, tries to get some leeway by subjecting to a newly-created theory. In all cases, it should also be noted, any change, deliberate or unintended, in the complex of our institutional subjections may lead us to experience unexpected setbacks.

How can we analyze the notion of a piece of knowledge $\mathscr{k}_I$? First point: knowledge is always knowledge *of something*, that is, of some *object* o, so that $\mathscr{k}_I$ should be rewritten $\mathscr{k}_I(o)$. Second point: except in the case of institutions with one position only, there is no such thing as what we have denoted by $\mathscr{k}_I$, $\mathscr{k}_\sigma$, $\mathscr{k}_\infty$, or $\mathscr{k}_\mu$. In reality, we must distinguish a priori between entities to be denoted generically by $\mathscr{k}_I(p, o)$. For example, in a school σ, we must (at least) distinguish between[7] the position s of student and the position d of teacher, so that we have (at least) two "pieces of knowledge," $\mathscr{k}_\sigma(s, o)$ and $\mathscr{k}_\sigma(d, o)$, that is, one "piece of knowledge for the student" and one "for the teacher."

We can now take a major step forward: the piece of knowledge denoted heretofore by $\mathscr{k}_I(p, o)$ is identified with the positional relation $R_I(p, o)$: $\mathscr{k}_I(p, o) \stackrel{\text{def}}{=} R_I(p, o)$. It results in particular from this definition that, in some cases, when the object o is unknown to the position p in I, i.e., when $R_I(p, o) = \varnothing$, then $\mathscr{k}_I(p, o) = \varnothing$: in other words, nothing is known of o to "pure subjects" of I in p—as such, they have to *not* acknowledge o.

Defining "knowledge" in terms of institutional relations to objects clearly extends the sense of the word well beyond its everyday usage—since dreaming about an object, for instance, is part and parcel of the "knowledge" one has of that object. However, there is more to it than that—for the new definition of knowledge raises some thorny questions. The first question we have to tackle boils down to this: If what has been denoted by $\mathscr{k}_I$ is the outcome of some transposition process, how does this translates in terms of relations $R_I(p, o)$? In order to arrive at an answer, we suppose the existence of a number of institutions J_l (with $l = 1, 2, \ldots, n$) in each of which there exists (at least) a distinguished position q_l such that $R_{J_l}(q_l, o) \neq \varnothing$. The creation in I of a relation $R_I(p, o)$, where p is an existing or yet to be created position in I, draws on some of the institutions J_l, chosen according to a variety of criteria—the ease of copying $R_{J_l}(q_l, o)$, the degree of prestige of the institution J_l being copied, the capacity of J_l to legitimize and defend the relation $R_I(p, o)$ developed in I, the relevance of $R_I(p, o)$ to the main activities going on in p, etc. In part, the elaboration of $R_I(p, o)$ after the relations $R_{J_l}(q_l, o)$ often behooves persons who have once been subjected to the positions q_l in some of the institutions J_l. However, more generally, by force of circumstance, this job is entrusted in I to persons until then foreign to the institutions J_l, who, starting from scratch, will have to *inquire* about the relations $R_{J_l}(q_l, o)$, taking into account a great many conditions that could be profitably satisfied by the relation $R_I(p, o)$ aimed at.

6 Also called "wild" children.

7 Here s stands for "student" and d for "docent"—the initial letter of *teacher* being reserved in ATD to denote a (particular) *task*, as we shall see in the next section.

It follows thereof that, between personal and institutional relations, there goes on an intricate interplay which contributes essentially to the creation, change, and demise of these relations. For every person x and every institution I and position p in I, let us define their *cognitive universe* by[8] $\Gamma(x) = \{(o, R(x, o)) \mid R(x, o) \neq \varnothing\}$ and $\Gamma_I(p) = \{(o, R_I(p, o)) \mid R_I(p, o) \neq \varnothing\}$, respectively. One can also define the cognitive universe of I by: $\Gamma(I) = \bigcup_p \Gamma_I(p)$, with p a position in I. The study of the genealogy of cognitive universes, both personal and positional, goes well beyond the sole analysis of didactic transposition processes. In what follows we shall focus succinctly on a partial aspect of such genealogical studies relating to personal and institutional relations. Let us define the *universe of objects* of a person x by: $\Omega(x) = \{o \mid R(x, o) \neq \varnothing\}$. Similarly, the universe of objects of a position p in an institution I is given by: $\Omega_I(p) = \{o \mid R_I(p, o) \neq \varnothing\}$, while the universe of objects of the institution I is $\Omega(I) = \bigcup_p \Omega_I(p)$, with p a position in I. The universe of objects of a person x or an institutional position p includes all the objects o known to x or p.

The hawk-eyed reader will have observed that, from a mathematical point of view, these definitions suppose the existence of a set $\check{U}$ of *all* the objects in the society $\check{S}$ we refer to. We should therefore have written: $\Omega(x) = \{o \in \check{U} \mid R(x, o) \neq \varnothing\}$, etc. In fact, in the foregoing developments we have feigned to ignore two fundamental aspects of the problem we are coping with—the diffusion of knowledge in $\check{S}$. The first aspect was however implicitly present: it is the passing of time, coextensive with the creation, change, and demise of cognitive matter. To be utterly rigorous, I should have written $\check{U}(\tilde{t})$ instead of $\check{U}$, $\Omega(\tilde{t}, x)$ instead of $\Omega(x)$, $R(\tilde{t}, x, o)$ instead of $R(x, o)$, and so on, where $\tilde{t}$ is the time. Except when necessary, I shall, however, continue to *not* mention time explicitly in the symbolic expressions used.

Another aspect has been left aside, which explains some essential (and sometimes irritating) features of institutions'—and therefore persons'—cognitive life. One can frame it as a puzzle: in the process of creation from scratch of a relation $R_I(p, o)$, how can the institution I know the existence of an object o until then utterly unknown to I? In some cases, it happens that each and every person x occupying one position or another in I knows o *personally*, while o is unknown to any position in I. In other words, for all position p in I, we have $R_I(p, o) = \varnothing$ and, for all x in position p, $R(x, o) \neq \varnothing$. The fact that, generally, $R(x, o) \backslash R_I(p, o) \neq \varnothing$ is a fundamental spring of the process by which an institution I can recognize an object o formerly foreign to it. The fact that, for at least some subjects x of I, we have $R(x, o) \neq \varnothing$ generally ensues from the fact that these subjects x are, or have been, subjected to a position q in an institution J where $R_J(q, o) \neq \varnothing$. If, moreover, it happens that, in one way or another, the institution J has sway over I, then the subjects of I will be tempted to copy J, even when such a transposition from J to I does not seem inevitable.

All this leads us to conclude that, in a transposition process from J to I, it can happen that, at the same time, a position q in J *and* the relation $R_J(q, o)$ are (partially) "copied"¸ thus giving birth in I to a position p and a relation $R_I(p, o)$ that conforms more or less to $R_J(q, o)$ and such that, in any case, $R_I(p, o) \neq \varnothing$. It follows that, even if o was not at first in the universe of objects of I, this is no longer the case—the object o being, from then on, known to I.

However another tricky question arise: how can two institutions I and J agree that they recognize the *same* object o? One sensible answer would be: they compare their respective institutional relations to the allegedly same object. To take just one example, provided I and J have the same notion of number, if both

8 The Greek capital letter Γ ("gamma") is the third of the Greek alphabet.

institutions can say that o is a number that, when multiplied by seven, gives three, they will concur to conclude that o is one and the same number—that you can write as the fraction 3/7. This easy example is typical of a great means of recognition: objects are labeled and these labels travel across institutions. For example, I and J may both speak of "fractions" long before they recognize that the objects thus labeled are not strictly identical. It is the dictionaries' job to record and itemize at least some of the different institutional meanings of a given "label." For example, *The American Heritage Dictionary of the English Language* acknowledges only one mathematical usage for "fraction"—a fraction is, it proclaims, "an expression that indicates the quotient of two quantities, such as 1/3." In the same way, the *Cambridge Advanced Learner's Dictionary* defines a fraction as "a number that results from dividing one whole number by another," and gives the following example: "1/4 and 0.25 are different ways of representing the same fraction." In contrast to these dictionaries, the *Collins English Dictionary* distinguishes two meanings: a) a fraction can be "a ratio of two expressions or numbers other than zero," and b) it can be "any rational number that is not an integer." This limited sample is enough to instantiate key phenomena affecting the circulation of labels. The third dictionary mentioned above records *two* definitions. The casual reader with a smattering of mathematics knows that the first is "elementary" while the second is more "advanced"—they refer to different (kind of) institutions and institutional knowledge. The first two dictionaries both restrict their focus to the elementary definition. But all three elementary definitions are plagued by a seeming contradiction. For the first dictionary, a fraction is not a number but[9] "an *expression* that indicates" a number—for example the expression $\frac{1}{3}$. For the second dictionary, a fraction *is* a number. But there's a snag! According to the same source, the "fraction" $\frac{1}{4}$ is *not* a number but a way "of *representing* a number"—which, in this particular case, is represented as well by the expression 0.25. The third dictionary blithely mingles *expressions* and *numbers*, leaving the question undecided. This apparent levity seems to be the rule rather than the exception in the cognitive universe to which these dictionaries refer. Let us consider one more example. The *Merriam-Webster's Online Dictionary* (11th Edition) offers the following "simple definition of *fraction*": "a *number* (such as 1/2 or 3/4) which *indicates* that one number is being divided by another," which combines the concepts of number and expression. Then it formulates this "full definition of *fraction*": "a numerical representation (as 3/4, 5/8, or 3.234) indicating the quotient of two numbers." The dominant meaning here is that of an expression that indicates something about the thing expressed—in the case at hand, a number. If we were to analyze more in depth the institutional relations that these dictionaries try to capture in a few words, we should certainly arrive at this conclusion: in the institutions which these definitions echo, there is a *potential* object lurking around, to wit, the fact that a symbolic expression both *designate* some mathematical entity and *shows* some of its properties—for example, the numerical expression $\frac{18+4}{24+5}$, which designates the number $\frac{22}{29}$, at the same time shows that we have $0.75 < \frac{22}{29} < 0.8$, provided we have available the following theorem: for any positive numbers a, b, c, and d, if $\frac{a}{b} < \frac{c}{d}$ then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. (Hint: think in terms of urns and balls.)

9 Here and in the following quotations, the emphasis is mine.

I will not go here into a deeper analysis of the institutional life of objects. I will nevertheless conclude this section with two more remarks. Firstly, it often happens that an object o lives permanently in an institution J and remains lengthily ignored by another institution I not unconnected with J, while being simultaneously unknown to the overwhelming majority of the persons subjected to I. Let me venture just one example, that of the mathematical notion of *elasticity*, a topic presently outside most secondary math curricula. For any function f of one real variable ξ defined and differentiable in a neighborhood of the point $a \in \mathbb{R}$, the elasticity at point a of f is defined as the limit $\mathrm{E}f(a)$ of the ratio of the relative changes of f and ξ with respect to the reference values $f(a)$ and a, respectively:

$$\mathrm{E}f(a) = \lim_{\xi \to a} \frac{f(\xi) - f(a)}{f(a)} \Big/ \frac{\xi - a}{a} = \lim_{\xi \to a} \frac{f(\xi) - f(a)}{\xi - a} \Big/ \frac{f(a)}{a} = \frac{a}{f(a)} f'(a)$$

Conversely, we have $f'(a) = \frac{f(a)}{a} \mathrm{E}f(a)$. The notion of elasticity of a function f is therefore no more sophisticated or miserable than the notion of derivative of f, which is central to the math curriculum, while the concept of elasticity remains ignored outside "math for economists" courses.

The preceding example illustrates a behavior widespread in institutions: the temptation of insularism, the propensity for "cognitive aloofness"—paradoxically associated with relations of dominance between institutions (and their corresponding relations to objects). In many cases, persons who come to be subjected to an institution I will discover an object o on that occasion for the first time, which means that o was hitherto unknown to all institutional positions to which they were previously subjected. Such a situation induces subjects of I to take for granted that o is an object specific to I—which, they feel, exists nowhere else. For example, it may happen that an Earth science teacher, who discovered the concept of "percolation" in the context of geology, has hardly ever heard of its many other uses—concerning, for instance, epidemic spreading through a population, or information spreading over networks. Such an illusive situation can in some cases be the result of deliberate maneuvers. Here is an example taken from a book published in the famous "For Dummies" series: *Physics Essentials for Dummies*, by Steven Holzner (2010, pp. 23-24). The problem to be solved consists basically in calculating the angle at the base θ and the hypotenuse h—which, concretely, is the distance from some point to a hotel entrance—of an isosceles right triangle whose side is 20 miles, using the Pythagorean theorem and some trigonometry. The hypotenuse h is given by $h = \sqrt{(20 \text{ mi})^2 + (20 \text{ mi})^2}$ ($= 20\sqrt{2}$ mi) $\approx$ 28,3 mi, so that the sine of the angle θ is given by the ratio 20/28.3, from which we get $\theta = \sin^{-1}(20/28.3) \approx 45°$. Rather unexpectedly, the author then concludes[10]: "You now know all there is to know. The hotel is 28.3 miles away, at an angle of 45°. *Another physics triumph!*" The mathematically naïve reader will be inclined to believe that Pythagoras' theorem and the inverse sine function are part and parcel of elementary physics!

All institutions tend to claim ownership of the objects that they rely upon—while ignoring all the others in $\check{U}$. This involves a subtle mechanism which can be summarized as follows. Firstly, there is a universal belief that any notion has a unique definition, independent of the institution that uses it. In other words, people adhere to a principle of uniqueness that applies to all entities they are interested in. (This principle is most surely an unconsidered extension of an axiom or principle regarded as true in physics—an atom is the

10 The emphasis is mine.

same "thing" everywhere in the whole universe.) Secondly, this postulate implies that the notion that inhabits "my" institution is exactly what this notion is, so that I can ignore all other institutions' definitions of it.

When one asks "What is a fraction?" the postulate behind the question is that there exists a univocal answer. Here is a typical question raised on a question and answer site ("How to explain," n.d.): "I found it difficult to explain the difference between the fraction a/b and the ratio a:b. This subject is for pupils of grade 5. So is there a real difference between them and how to explain the difference in simple way?!" Here it is supposed, not only that the expressions a/b and a:b designate distinct (mathematical) realities, but also that their meanings are uniquely determined. In contrast with this immutability belief, the ATD-didactician, enlightened by the didactic transposition theory, will question all objects whatever in terms of institutional place I and time $\tilde{t}$—for example the question "What is a fraction?" will be rephrased as "How, in the position p of the institution I, was the notion of fraction defined at the time $\tilde{t}$ of I's history?" Such an inquiry leads to the study of the genealogy and "archeology" of objects, which paves the way for a didactic science of knowledge and knowing.

4. THE THEORY OF PRAXEOLOGIES

The theory of praxeologies is a subtheory of ATD that has been developed as an answer to the question: Where do personal and institutional relations come from? In other words, how can we "explain" the content of the personal relation $R(x, o)$? How come it includes this but not that? The first remark on which an answer can be based is that our cognitive universe is the result of our activity: I came across the object o because I had to do something in which o played some role or other. What I had to do, what I do—handle something, think about it, love it, discard it, etc.—determines my (personal) relation to the objects that make up my cognitive universe. The notion of a praxeology was introduced as an essential means of analyzing human activity—be it mathematical or otherwise. This is where the reason for labeling "anthropological" the theory developed is most obvious.

The starting point is the (anthropological) postulate that any activity α conducted in an institution I splits into a number of basic "parts" called *tasks*, a fact that we shall write: $\alpha = t_1 \wedge t_2 \wedge \ldots \wedge t_n$. Every task t_i is of a certain *type* T_i, which is generally expressed by a verb of action with a direct object, like "to work out a multiplication," "to write a novel," "to shoot a movie," "to watch TV," "to cook an omelet," "to drive a car," "to sing a song," "to solve a quadratic equation," "to welcome a guest," "to hurtle down a flight of stairs," "to divide two fractions," "to dye one's hair," "to blow one's nose," "to calculate a definite integral," "to add two fractions," "to ask someone to dance," "to start to compose an opera," "to establish the exact algebraic expression of the area of a regular heptagon," "to look after the children," etc. Three points deserve to be emphasized. Firstly, the notion of *type of tasks* in ATD applies to *any kind* of human activity—be it mathematics, cooking, daily body care, entertainment, literature, artistic creation, education, or what have you. Secondly, a type of task can be something "small" ("to open a can," "to blow one's nose") or something much "bigger" ("to write a three-volume treatise on ATD," "to run for president of the United States of America"): once again, size does *not* matter. Thirdly, the notion of task in ATD has little to do with the notion of "a piece of work" in the ordinary sense of this expression—"to scratch one's ear" is generally *not* considered labor. Moreover, the boundary between work and nonwork is fragile: if you go to the kitchen to fetch a new glass of water, you will not count it as work, unless you are a salaried maid and have this task imposed on you. (There is an old joke, now probably considered sexist, and often attributed to the British economist John Maynard Keynes (1883-1946), that if a man marries his maid, then the GNP drops, because domestic services that were performed by a salaried maid are now performed for free.) Yet another remark should be made. It concerns the distinction between a *type* of tasks T and a particular *task* t of this type. A task $t \in T$ is called a *specimen* of T. A specimen, dictionaries say, is "an example regarded as typical of its class." Here the word is taken to simply mean a member of T—for, on approaching T for the first time, we generally do *not* know to what extent a given "specimen" t is "typical" of T.

Let us note that the tasks and types of tasks are objects in $\check{U}$—the set of all objects of the society $\check{S}$. More precisely, whenever T exists for the position p of the institution I, it belongs to the sets $\Omega(I) = \bigcup_u \Omega_I(u)$ and $\Omega_I(p) = \{o \mid R_I(p, o) \neq \emptyset\}$. But the existence of any of these objects o_i depends on the existence (for the position p) of a great many objects o_{ij}—in the examples above, at least the objects "natural number," "multiplication," "novel," "movie," "TV," "cooking," "car," "song," "quadratic equation," "guest," "flight of stairs," "division," "dying," "integral," "dance," "opera," "regular heptagon," and so on and so forth. As a general rule, the coming to life of any type of tasks T entails the availability of a great many objects, prior to defining T. Such an organized complex of objects form what will be called the *infrastructure* of T. As we shall see, the notion of infrastructure is, in ATD, a general concept: it refers to the underlying base needed to develop any determined reality. One easily imagines that "watching TV" at home, for example, requires an infrastructure that might seem extravagant to the uninitiated.

Now to the second postulate of the theory of praxeologies. In essence, this postulate holds that, in any institutional position p where tasks of a type T have to be performed on a regular basis, there develops some "way of doing" tasks of type T, which, in the medium run, will be institutionalized. It is such a "way of doing" a task of type T that we call a *technique* for the type of tasks T and denote generically by the Greek letter[11] τ (tau)—rigorously we should write it τ_T. Let me highlight that it is a tenet of ATD that *any* type of tasks T, no matter how trifling or insignificant it may be—think of scratching yourself behind the ear!—, requires (at least a sketch of) a technique τ in order to carry out specimens of this type.

Suppose, for example, that you want to calculate 5% of 3400, a task which is a specimen of the type of tasks "Calculate a% of b." To do so, you can determine how many hundreds are in 3400, and then multiply the number found by 5, that is, work out the product of 5 and 34. Calculating mentally or using a calculator, you thus arrive at: 5% of 3400 = 170. Note that, in this technique, the number of "hundreds contained" in the number b is the quotient of b by $a = 100$, that is to say, b/a (and not $b\backslash a$, that is, the integer quotient). If $b = 3427$, the number of hundreds "contained in" b must be understood to be the *decimal* number 34.27 (and not the whole number 34). The percentage we are after is therefore equal to 5×34.27, i.e., 171.35. In this case, the mathematical infrastructure drawn upon must include decimal numbers and their multiplicative structure—which, historically speaking, is quite a mathematical gem. Note that we could also get the result by submitting the following straightforward query to Google's search engine[12]: ⟨5% of 3427⟩. (The answer

---

11 The letter τ is the 19th letter of the Greek alphabet.

12 The search query is the text between the left and right angle brackets ⟨ and ⟩.

given by this search engine is, of course, 171.35.) We can also use the formula $a\%$ of $b = \frac{a \times b}{100}$, which gives: 5% of $3427 = \frac{5 \times 3427}{100} = \frac{17135}{100} = 171.35$. Note that, in the present case, this technique of calculating percentages allows us to shun the multiplication of decimals.

Suppose now you want to find the integer quotient of, say, 319 by 12. This a specimen of the type of tasks that consists in determining the quotient in a Euclidean division. In the case at hand, one possible—though unfamiliar—technique is to look for the successive quotients of 319 by 2, 2, and 3—since we have: $12 = 2 \times 2 \times 3$. The integer quotient of 319 by 2 is equal to the (integer) quotient of 318 by 2, which is 159. The integer quotient of 159 by 2 is equal to the integer quotient of 158 by 2, which is 79. The integer quotient of 79 by 3 is equal to the quotient of 78 by 3, which is 26. *Therefore*, the integer quotient in the Euclidean division of 319 by 12 is 26. (We have indeed: $319 - 12 \times 26 = 319 - 312 = 7 < 12$.) If you allow yourself to draw on today's infrastructural resources, you can as well use a calculator to get the answer: we have[13] $319/12 =_c 26{,}5833\ldots$, therefore the integer quotient of 319 and 12 is 26: $319\backslash 12 = 26$. Some readers may wonder if this latter "technique," denoted by $\tau_\backslash$, is indeed sound: we shall return to this point a little further on.

The above examples suffice to point out a number of key facts about techniques. Firstly, to carry out a technique τ amounts to performing a succession of tasks $t_1 \wedge t_2 \wedge \ldots \wedge t_n$. In turn, in performing the tasks $t_i \in T_i$ one must resort to techniques τ_i relating to the type of tasks T_i. In turn again, the techniques τ_i lead to performing a succession of tasks $t_{i,1} \wedge t_{i,2} \wedge \ldots \wedge t_{i,ni}$; and so on. Thus there exists a dialectical interplay between techniques and types of tasks. One important consequence of this interplay lies in the fact that, thanks to an appropriate choice of τ, it is sometimes possible to avoid having 1) to use infrastructural resources not currently available and 2) to perform a task of a type considered as practically beyond current reach.

Another general fact that should not go unnoticed is linked to the notion of *scope* T_τ of a technique τ for a type of tasks T: the scope T_τ of τ is the (fuzzy) subset of T made up of the tasks $t \in T$ that the technique τ allows one to successfully carry out. Now the fact that must be stressed is that no technique τ allows us to carry out *all* the tasks $t \in T$. In other words, we always have $T_\tau \subsetneq T$. In this respect, it is enough to observe that most calculation techniques do not succeed when the size of operands increases immensely. For example, familiar techniques do not allow to factor into primes, within a human lifetime, most 10^4-digit integer—even if, as we all know, numbers of the same length may prove unequally hard to factor.

When an institution I considers endowing one of its position p with a technique τ, I has to take into account several factors. One of them is certainly the fact that the tasks t of type T performed in p (almost) all belong to T_τ. Let me take an example. Suppose we consider that two variables u and v are proportional: $v \propto u$. Suppose now that, if $u = 7$, then $v = 259$. What if $u = 14$? Most people will answer without further ado that v will be *two times* 259, because 14 is two times 7. In like manner, we will rejoin that if $u = 21$, then $v = 3 \times 259$ (= 777), etc. But what if, say, $u = 17$ or 23? It seems that these cases (in which u is not divisible by 7, i.e., is not a multiple of 7) fall *outside* the scope of this common-sense technique—which, consequently, appears to be marred by a strangely narrow scope. Another possible technique, with a wider scope, is the

13 The symbol $=_c$ refers to the value displayed by a calculator.

so-called *unitary method*, which, traditionally, is regarded as a technique for the lightly educated: if, when $u = 7$, we have $v = 259$, then we can conclude that, if $u = 1$, v will be *seven times less*, that is, $v = 259/7 = 37$; therefore, when $u = 17$, we will have $v = 17 \times 37$ (= 629). The unitary technique is clearly an improvement of the common-sense method—if, for example, $u = 23$, then v is *twenty-three times* the value of v for $u = 1$, that is 23×37 (= 851).

We can however improve on the unitary method. Suppose the mathematical infrastructure we may call the *arithmetic of fractions* is available. The common-sense technique can then be recast as follows. When $u = 14$, we write $\frac{259}{7} = \frac{2 \times 259}{2 \times 7} = \frac{518}{14}$, which gives the value of v when $u = 14$; when $u = 21$, we have $\frac{259}{7} = \frac{3 \times 259}{3 \times 7} = \frac{777}{21}$ and reach an analogous conclusion. What, then, if $u = 17$ or 23? We have: $\frac{259}{7} = \frac{\frac{17}{7} \times 259}{\frac{17}{7} \times 7} = \frac{\frac{17}{7} \times 259}{17}$. Therefore, when $u = 17$, we have $v = \frac{17}{7} \times 259$. We can interpret this calculation in terms of a natural generalization of the notion of a *whole* "number of times": the notion of a *fractional* number "of times." For example, $u = 23$ is $\frac{23}{7}$ times 7, so that v will be $\frac{23}{7}$ times 259 or $\frac{23}{7} \times 259$. To conclude, we can use a calculator: if we submit to Google's search engine the queries ⟨(17 / 7) * 259⟩ and ⟨(23 / 7) * 259⟩, the search engine returns the values 629 and 851, respectively, as expected. More generally, given a pair (u_0, v_0) and a value u, this enriched technique can be written thus: $\frac{v_0}{u_0} = \frac{\frac{u}{u_0} \times v_0}{\frac{u}{u_0} \times u_0} = \frac{\frac{u}{u_0} \times v_0}{u}$. Note that this "fractionally enriched" technique allows us to ignore whether or not u is a multiple of u_0. If, for example, $u = 1029$, one can conclude that v will be $\frac{1029}{7}$ times $v_0 = 259$, thus arriving at $v = \frac{1029}{7} \times 259 =_c 38073$. In fact, $1029 = 147 \times 7$ and, as expected, $147 \times 259 =_c 38073$.

The preceding example points to a massive, key fact. Mathematical infrastructures tend to enable one to resort to techniques with a wider scope, that, at the same time, are more powerful and often easier to perform. Here is a new example, this time from elementary geometry. In the plane endowed with a Cartesian coordinate system, we consider the three points A (8, 0), B (–4, 4), and C (0, –4). When one looks at the figure below, it seems that ABC is an isosceles right triangle with right angle in C.

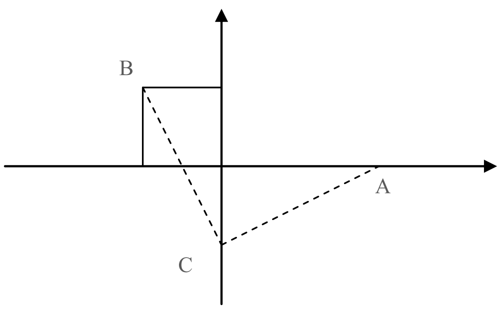

Figure 1. An isosceles right triangle?

How can we verify this conjecture? Any technique can be analyzed as specifically composed of an "infrastructural" part—an "apparatus"—on which the user performs a "superstructural" activity by making specific "gestures." Here as elsewhere, the mathematical work to be done is thus a superstructural performance that arises on an infrastructural base whose construction is more or less demanding. In the present case, let us suppose that we have available the arithmetic of *complex* numbers—an infrastructural achievement that generally requires a massive effort. The superstructural work needed in this case is, by contrast, an easy-going affair. The points A, B, and C are the affixes of the complex numbers $a = 8$, $b = -4 + 4i$, and $c = -4i$. The affixes of vectors $\overrightarrow{\mathrm{CA}}$ and $\overrightarrow{\mathrm{CB}}$ are $z_{\mathrm{A}} = a - c$ and $z_{\mathrm{B}} = b - c$. Our conjecture is therefore equivalent to: $\pm iz_{\mathrm{A}} \stackrel{?}{=} z_{\mathrm{B}}$. We have $z_{\mathrm{A}} = 8 - (-4i) = 8 + 4i$ and $z_{\mathrm{B}} = (-4 + 4i) - (-4i) = -4 + 8i$, so that $iz_{\mathrm{A}} = i(8 + 4i) = -4 + 8i = z_{\mathrm{B}}$. Our conjecture is proved. As a general rule, the choice of a technique τ is a trade-off between the effectiveness and smoothness of use of τ, on the one hand, and the cost of the infrastructure it rests on top of, on the other hand.

The ordered pair of a type of tasks T and a technique τ for performing tasks $t \in T$ is denoted by[14] $\Pi = [T / \tau]$ and is called a *praxis block*—from the ancient Greek word *praxis* meaning "practice, action, doing." It is the "know-how" component of "knowledge," the mastery of which can be called a *skill*. An important aspect of a *praxis* block $\Pi = [T / \tau]$ is the "trustworthiness" of τ, that is to say the fact that, all other conditions held constant, τ lends itself little to mistakes—in other words, though not error-proof, τ is a "surer" technique, as far as tasks $t \in T_\tau$ are concerned. Now one major fact is that the safety of use of a technique τ increases with its "intelligibility" as experienced by the user. As a way of doing something, a technique is a safer one if it is "understandable" by its average user—the generic subject in the position p in I—who then easily makes sense of it.

This remark has far-reaching implications. Anthropologically, that is, as *Homo sapiens*, we are inclined "to ask why"—about any matters. To understand a given technique τ is to understand *why* τ requires to perform the task t_1 (using the technique τ_1), then the task t_2 (adopting the technique τ_2), and so on—we suppose here that performing τ on t equates to performing the succession of tasks $t_1, t_2, \ldots t_n$. To understand (why) τ amounts to understanding why performing $t_1, t_2, \ldots t_n$ add up to performing t. To this need responds in the position p of I a "discourse" on τ that purports to *explain* τ, that is an account of τ that claims to "make it clear." It is such a discourse, that can vary from institution to institution, and even from position to position within a given institution, that is called a *technology* of the technique τ and is denoted by the Greek letter[15] θ (theta). The aim of a technology θ is to make the technique τ intelligible, to explain why it is what it is—and why it is as it is and not otherwise—even if there exist "competing" techniques τ' for T.

A weak but important form of "explanation" is *justification*. Originally, this word has to do with "justice": to justify a technique τ is a way to do justice to it by showing that τ does precisely what it is supposed to do. The notion of justification may well be different according to the institution I and position p. As we all know, in mathematical institutions, it takes the form of a so-called *proof*, a notion that, since the beginning of "formal mathematics" until today, has always been in the making. Let us go back to the technique $\tau_\backslash$ for calculating the integer quotient of two whole numbers. We have for example: 1049\30 =

14 The Greek capital letter Π ("pi") is the 16th letter of the Greek alphabet.
15 The Greek small letter θ ("theta") is the 8th letter of the Greek alphabet.

$1049\backslash(2 \times 3 \times 5) = 1045\backslash(2 \times 3 \times 5) = 209\backslash(2 \times 3) = 208\backslash(2 \times 3) = 104\backslash 3 = 102\backslash 3 = 34$. (The remainder is $1049 - 30 \times 34 = 1049 - 1020 = 29 < 30$.) Note first that the use of the backslash (or reverse solidus) symbol, as indicated above, proves greatly helpful: by enriching our notational infrastructure it allows us to recast the technique $\tau_\backslash$ to get a less lead-footed, more easily monitored, and therefore less error-prone technical variant $\tau_\backslash'$.

I want now to stress my main point as regards $\tau_\backslash$ (or $\tau_\backslash'$): to a great number of mathematics teachers I know, this technique will appear as counterintuitive and, in fact, unjustified—even if the two numerical examples considered so far fail to disprove it. Let us therefore *prove* it—in the mathematical sense of the term. Let $a, b \in \mathbb{N}$ with $ab \neq 0$. It should be clear that the whole number $q = a\backslash b$ is the unique integer such that $q \le a/b < q + 1$, i.e., such that $bq \le a < b(q + 1)$, or, still equivalently, such that $bq \le a$ and $a + 1 \le b(q + 1)$. Let us further suppose that $b = b_1 b_2$ (with $b_1, b_2 \in \mathbb{N}$). We set $a\backslash b_1 = q_1$ and $q_1\backslash b_2 = q_2$. Let us prove that $q_2 = a\backslash b$. On the one hand, we have $b_1 q_1 \le a$ and $b_2 q_2 \le q_1$, so that $bq_2 = b_1 b_2 q_2 \le b_1 q_1 \le a$. On the other hand, we have $a + 1 \le b_1(q_1 + 1)$ and $q_1 + 1 \le b_2(q_2 + 1)$, so that $a + 1 \le b_1 b_2(q_2 + 1) = b(q_2 + 1)$. This allows us to conclude that $q_2 = a\backslash b$. The theorem is proved.

The proof given here is typical of what could be found in the literature many decades ago—when the technique $\tau_\backslash$ used to be taught at the lower secondary level. As a technological discourse, however, it is far from satisfactory: it proves that the technique $\tau_\backslash$ does work well indeed, but it fails to help us understand *why* it works. In other words, it is an incomplete technology of $\tau_\backslash$. It is a mathematically highly-valued but cognitively minimalistic "explanation"—a fact both vindicated by mathematical tradition—for which a proof should first of all be "irresistible" (Pascal, 1909-1914) to quote Blaise Pascal (1623-1662)—and recognized by popular culture as expressed by the phrase "to prove by A plus B," or, to quote *Candide* (1759/1901) by Voltaire (1694-1778), "by A plus B minus C divided by Z."

An optimal technology should both be a proof of the correctness of $\tau_\backslash$ and make clear why it is correct. Let's have a try at it. Suppose again that $b = b_1 b_2$ (with $b_1, b_2 \in \mathbb{N}$). The quotient $q = a\backslash b$ is such that $a = bq + r$ with $0 \le r < b$. Before going any further, let me introduce a "concrete" model of the mathematical situation under examination. The rationale for this model is the following. Let us consider a "things," for example a chocolates. The equality $a - r = bq$ means that $a - r$ of them form a set s that can be regarded as the union of q b-sets s_i ($1 \le i \le q$), each of which is the union of b_2 sets t_{ij} of b_1 chocolates: $s = \bigcup_{1\le i\le q} s_i = \bigcup_{1\le i\le q} (\bigcup_{1\le j\le b_2} t_{ij})$: we have $|t_{ij}| = b_1$, $|s_i| = b_1 b_2$, and $|s| = qb_1 b_2 = qb$. Now Let $q_1 = a\backslash b_1$, so that we have $a = b_1 q_1 + r_1$ with $0 \le r_1 < b_1$. Suppose you have a chocolates and want to make *bags* of b_1 chocolates, as shown on figure 2, where triangles stand for chocolates, with $b_1 = 8$.

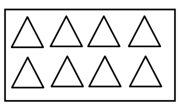

Figure 2. A *bag* of chocolates

In the case considered here, we can write: $a = 8q_1 + r_1$, with $0 \le r_1 < 8$. We can therefore fill q_1 bags with $b_1 = 8$ chocolates with, say, $r_1 = 7$ remaining chocolates. We shall soon see that these remaining chocolates can be "jettisoned" because they will not be "utilized" any longer in this division process—even if the chocolates left over from this operation will certainly not be lost for everyone! Let us continue the division and its concrete modeling. Suppose we want to fill *boxes* of chocolates with $b_2 = 6$ bags of chocolates, as suggested by the figure below.

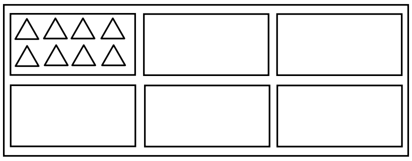

Figure 3. A *box* (of bags) of chocolates

The number of boxes that we can make is therefore $q_2 = q_1 \backslash b_2$. We have: $q_1 = b_2q_2 + r_2$, with $0 \leq r_2 < b_2$. Suppose for example that $r_2 = 5$. With the 5 remaining bags we cannot make one more box, for one bag (= 6 – 5) is wanting and the 7 remaining chocolates cannot help! It should now be obvious that the leftovers consisting of the $r_1 = 7$ chocolates and the $r_2 = 5$ bags (of 8 chocolates each) can be safely overlooked.

Let us now *prove* this result in a way inspired by our concrete model. We have $a = b_1q_1 + r_1$ with $0 \leq r_1 < b_1$ and $q_1 = b_2q_2 + r_2$, with $0 \leq r_2 < b_2$. We get: $a = b_1q_1 + r_1 = b_1(b_2q_2 + r_2) + r_1 = b_1b_2q_2 + b_1r_2 + r_1 = bq_2 + (b_1r_2 + r_1)$. The value of $b_1r_2 + r_1$ in our example is $8 \times 5 + 7 = 47$, which is indeed strictly less than $b = b_1b_2 = 8 \times 6 = 48$. More generally, we have: $b_1r_2 + r_1 \leq b_1(b_2 - 1) + (b_1 - 1) = b_1b_2 - b_1 + b_1 - 1 = b_1b_2 - 1 < b_1b_2$. Therefore $a \backslash b = q_2$. ■

Given *any* praxis block $\Pi = [T / \tau]$ and a position p in an institution I where Π has come to be lodged, we usually observe the association of Π with a technology θ that justifies τ and makes it intelligible to users—i.e., to the persons occupying the position p in I. The example of $\tau_\backslash$ shows that the lack of an appropriate technology can jeopardize τ and even cause its disappearance from the institutional scene. However, another phenomenon often occurs: the "fabrication"—endogenous to the position p or the institution I—of the lacking technology. As a result, people in p will often propagate questionable claims about τ. For example, it is well known that, in lay culture, people often believe that, if the price of a commodity increases by 10% and then again by 10%, this price will have increased by 20%, while it has really increased by 21%. As is well known, too, this spontaneous, easy-going technique is induced by a popular, widespread, *additive* technology which has the charm of mathematical simplicity, while the mathematically correct technique rests on a mind-bending *multiplicative* model—if a quantity q increases by a% and then again by b%, it eventually increases by $(a + b + \frac{ab}{100})$% because the new quantity q' is given by $q' = q(1 + \frac{a}{100})(1 + \frac{b}{100}) = q(1 + \frac{a+b}{100} + \frac{ab}{10\,000})$, so that the increase is $(a + b + \frac{ab}{100})$% of the quantity q.

It would be erroneous to believe that "false beliefs" are the preserve of lay-world technologies. During the greater part of the nineteenth century, the French secondary math curriculum entirely ignored calculus. To solve problems of maxima and minima, students were initiated to an algebraic technique whose key technological backing was (and still is) the following theorem, that I will call here Cauchy's theorem because it behooved Augustin Louis Cauchy (1789-1857) to give the first flawless proof of this statement in his *Cours d'analyse* (Cauchy, 1821, pp. 457-459; Bradley & Sandifer, 2009, pp.306-307): the product of n positive numbers with constant sum is largest when all the numbers are equal. This theorem and its dual statement—"the sum of n positive numbers with a constant product is minimal when the numbers are equal"—permit to solve most elementary problems of maxima and minima (Niven, 1981; Rike, 2002). Let us consider for example the triangles with a given perimeter. Which of them has the largest area? The area of a triangle with side lengths a, b, and c is given by Heron's formula[16]: $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi-perimeter $\frac{a+b+c}{2}$, which is also a constant. Maximizing Δ amounts to the same thing as maximizing $\Delta^2 = s(s-a)(s-b)(s-c)$, which in turn is the same as maximizing the product $(s-a)(s-b)(s-c)$. The sum of the three factors of this latter expression equals s and is therefore constant. According to Cauchy's theorem, Δ is a maximum when $s-a = s-b = s-c$, which gives $a = b = c$, as expected.

In the absence of calculus, this elementary technique was exclusively used in French senior high schools. But the technology proffered in textbooks had a big flaw. The gist of the proof that was then traditional was that, if the product $a_1a_2a_3 \ldots a_n$ has two *unequal* factors, for example a_1 and a_2, then that product cannot be a maximum. Let us set $a'_1 = a'_2 = \frac{a_1+a_2}{2}$. We have $a'_1 + a'_2 = a_1 + a_2$ and $4(a'_1a'_2 - a_1a_2) = (a_1+a_2)^2 - 4a_1a_2 = (a_1-a_2)^2 > 0$, which proves the theorem. But the theorem thus "proved" says only that, *if* a maximum does exist, *then* it occurs when all the factors $a_1, a_2, a_3, \ldots, a_n$ are equal. Since the time of Jakob Steiner (1796-1863), most proofs of isoperimetric problems (like the problem about triangles above) were spoiled by this error, against which Oskar Perron (1880-1975) eventually raised what is known today as "Perron's paradox": "Let n be the largest integer. If $n > 1$, then $n^2 > n$, contradicting the definition of n. Hence $n = 1$."

Instead of going deeper into this aspect—the everlasting battle of "true knowledge" against "obvious" but fallacious arguments—, I want to especially stress the problem raised by the absence of a positionally (or personally) "appropriate" technological discourse on a given technique τ. Such absence leads to the spontaneous production and propagation of "idiosyncratic" ideas and beliefs acting as an ersatz technology: when deprived of a suitable technology, all institutions, all persons speedily create their own technology—most often by tinkering with pieces of technology from other institutions. Now this phenomenon is greatly governed by "general" ideas about the domain of activity involved. As concerns for example the problem of the maximum of a numerical product whose factors add up to a constant number, and more generally regarding isoperimetric problems, the general belief seems to have obscurely, but rightly, anticipated the *extreme value theorem* for continuous functions on compact sets ("Extreme value theorem," n.d.). Many memorable episodes in the history of mathematics remind us of the fragility of the best mathematical minds faced with hitherto unproblematized situations. For example, in a letter to Richard Dedekind (1831-1916), Georg Cantor (1845-1918) proved in 1877 the existence of a bijection between $[0, 1]$ and $\mathbb{R}^n$ for all positive integer n. About this unexpected discovery and his proof of it, Cantor wrote to Dedekind in French (Gouvêa, 2011): *Je le vois, mais je ne le crois pas* ("I see it, but I don't believe it"). Here again, the correct mathematical explanation of Cantor's unquestionable finding would not be ascertained before some time[17].

16 The Greek capital letter Δ is the 4th of the Greek alphabet.
17 For a slightly different construal of Cantor's ecphonesis, see, however, Gouvêa (2011).

One could add many more examples of such a "defeat of intuition." Let me simply recall that the French mathematician Henri Poincaré (1854-1912) once wrote (Poincaré, 1907):

> We know there exist continuous functions lacking derivatives. Nothing is more shocking to intuition than this proposition which is imposed upon us by logic. Our fathers would not have failed to say: "It is evident that every continuous function has a derivative, since every curve has a tangent." (p. 17)

In all the cases examined up to now, the key factor lies in a number of ideas and notions generally common to the persons subjected to the same institution or position, that impose or, at least, fashion the technological elements these persons refer to in relation to the tasks they perform and the roles they fulfill. These ideas are located at a higher level in the organization of human activity—the level of *theory*. In ATD, a theory is a "discourse" (in an extended sense of the word—such a discourse may include symbols, calculations, diagrams, etc.), generally denoted by the Greek capital letter Θ, that can generate, control, justify, and make intelligible a given technology (or a bundle of technological discourses).

Two points should be emphasized. Firstly, the notion of theory that is specific to ATD subsumes the sense of theory habitually used in science: the theoretical part of a scientific construct does fall under this notion of theory, although what is called theory in ATD generally *exceeds* what "scientific insiders" look at as theory—a mathematical (or physical, or psychological, etc.) theory in the sense of ATD contains by necessity other unanalyzed "anthropological"—i.e., nonmathematical (or not pertaining to physics, to psychology, etc.)—elements, that mathematicians (or physicists, or psychologists, etc.) generally ignore, consciously or not. Secondly, and this may be more difficult to receive, for any triplet formed of a type of tasks T, a technique τ, and a technology θ, there will exist an associated theory Θ. This tenet implies, for example, that when one brushes one's teeth, this person's behavior is partly determined by the theoretical "ideas" the person holds about dental hygiene, and these ideas may in turn be determined by the person's relations to *many* objects (toothbrush and toothpaste, obedience to parents in the case of a young child, a sense of the fragility of life, daily rituals, etc.). As a rule, a theory Θ in the sense of ATD has thus to do with a host of objects which, through the intermediary of praxeologies, contribute to shape persons' and institutions' behavior.

The use of "theory" in ATD recoup familiar connotations of this word: *theory* still carries the idea of a distanced stance, that of a spectator—the meaning of Greek *theoros*—toward the activity domain the theory in question is concerned with. In his *Dictionary of Words Origins* (1990), John Ayto writes pertinently:

> **theory** [161 The etymological notion underlying *theory* is of 'looking'; only secondarily did it develop via 'contemplation' to 'mental conception'. It comes via late Latin *theōria* from Greek *theōría* 'contemplation, speculation, theory'. This was a derivative of Greek *theōrós* 'spectator,' which was formed from the base *thea-* (source also of *thesthai* 'watch, look at,' from which English gets *theatre*). Also derived from *theōrós* was *theōren* 'look at,' which formed the basis of *therēma* 'speculation, intuition, theory,' acquired by English via late Latin *theōrēma* as *theorem* [16]. From the same source comes *theoretical* [17]. (p. 527)

The association of a technology θ and a theory Θ constitutes a *logos block*, which is denoted as follows[18]: $\Lambda = [\theta / \Theta]$. The Greek word *logos* used here originally meant "word, speech, statement, discourse." John Ayto (1990) writes at the entry "logarithm":

> Greek *lógos* had a remarkably wide spread of meanings, ranging from 'speech, saying' to 'reason, reckoning, calculation,' and 'ratio.' The more 'verbal' end of its spectrum has given English the suffixes *–logue* and *–logy* (as in *dialogue*, *tautology*, etc.), while the 'reasoning' component has contributed *logic* [14] (from the Greek derivative *logiké̄*), *logistic* [17] (from the Greek derivative *logistikós* 'of calculation'), and *logarithm*, coined in the early 17th century by the English mathematician John Napier from Greek *logós* 'ratio' and *arithmós* 'number' (source of English *arithmetic* [13]). (p. 328)

The *logos* block corresponds to what most people have in mind when they use the term "knowledge"—although one can reasonably argue that knowledge is the dialectical union of *logos* and *praxis*. In fact, the *logos* block $\Lambda = [\theta / \Theta]$ that is normally associated with the *praxis* block $\Pi = [T / \tau]$ is "amalgamated" with it to constitute a *praxeology*, as shown here:

$$\Pi \oplus \Lambda = [T / \tau] \oplus [\theta / \Theta] = [T / \tau / \theta / \Theta].$$

It is the quadruplet $\wp = [T / \tau / \theta / \Theta]$ that is called *a praxeology*. The arborescent structure of human activity—resulting from the interplay, sketched above, between types of tasks and techniques—suggests that any person x or institutional position p uses a multitude of praxeologies. These praxeologies compose the person x's *praxeological equipment* $E(x)$ or the position p's *praxeological equipment* $E_I(p)$.

The relation of x to an object o, $R(x, o)$, derives from the repertoire of praxeologies $\wp \in E(x)$ that, in one way or another, "activate" the object o. The same applies to an institutional position: the relation $R_I(p, o)$ emerges from the changing set of praxeologies involving o that are activated by the subjects of I occupying the position p. This is well illustrated by the case of the "forgetful" technique of division $\tau_\backslash$. Where does the resistance to this "peculiar" technique come from? The problem, for sure, is with the deliberate omission of numbers that are spontaneously regarded as "remainders"—in fact, they *are* remainders. The habit developed at school with traditional division algorithms to take great care of "remainders" leads one to feel ill at ease with "throwing away" these particular remainders. Let us indulge one more time in a "forgetful" division. With 319 as dividend and 12 as divisor, we have: 319\12 = 318\12 = 159\6 = 158\6 = 79\3 = 78\3 = 26. However, using the familiar division sign, we can rewrite this succession of equalities in an "old-style", enlightening fashion: 319/12 = 318/12 + 1/12 = 159/6 + 1/12 = 158/6 + 1/6 + 1/12 = 79/3 + 1/6 + 1/12 = 78/3 + 1/3 + 1/6 + 1/12 = 26 + 1/3 + 1/6 + 1/12. We have: 1/3 + 1/6 + 1/12 = 7/12 < 1, which proves that 319\12 = 26. This (numerical) observation can be easily generalized to get a full-fledged proof. Let us consider the case in which b is the product of *two* positive integers b_1 and b_2. Here again, q_1 and r_1 are the quotient and remainder in the division of a by b; and q_2 and r_2 are the quotient and remainder in the division of q_1 by b_2. By following the preceding numerical example, we can arrive at this[19]:

$$\frac{a}{b} = \frac{a}{b_1b_2} = \frac{b_1q_1}{b_1b_2} + \frac{r_1}{b_1b_2} = \frac{q_1}{b_2} + \frac{r_1}{b_1b_2} = \frac{b_2q_2 + r_2}{b_2} + \frac{r_1}{b_1b_2} = \frac{b_2q_2}{b_2} + \frac{r_2}{b_2} + \frac{r_1}{b_1b_2} = q_2 + \frac{r_2}{b_2} + \frac{r_1}{b_1b_2} = q_2 + \rho_2$$

18 The Greek capital letter Λ—the 11th letter of the Greek alphabet—is the initial of *logos* (λόγος).

19 The Greek small letter ρ is the the 17th letter of the Greek alphabet.

with $\frac{r_2}{b_2} + \frac{r_1}{b_1 b_2}$. We then have:

$$\rho_2 \leq \frac{b_2 - 1}{b_2} + \frac{b_1 - 1}{b_1 b_2} = 1 - \frac{1}{b_2} + \frac{1}{b_2}\left(1 - \frac{1}{b_1}\right) = 1 - \frac{1}{b_2} + \frac{1}{b_2} - \frac{1}{b_1 b_2} = 1 - \frac{1}{b_1 b_2} < 1.$$

It should be obvious that, more generally, when $b = b_1 b_2 \ldots b_n$, we have: $\frac{a}{b} = q_n + \rho_n$ with $\rho_n \leq 1 - \frac{1}{b_1 b_2 \ldots b_n} < 1$. The theorem is proved—once again.

In the case of the "forgetful" division algorithm, three aspects can be highlighted to provide greater clarity on the ecology and economy of the *logos* block. The first point that I want to emphasize is that the *logos* block really bears on the *praxis* block, to the point of nipping in the bud a potential technique. Second aspect: our example shows us that, even in mathematics, the technology of a technique cannot be reduced to just a proof of the technique: it has also to explain it and to make it intelligible. In this respect, our treatment of the forgetful algorithm suggests that, contrary to an oversimplification present in school mathematical culture, what ATD calls technology can hardly consist in just one, well-chosen proof: it often necessitates *several* proofs, from different points of view, that serve not only to confirm the "inaugural" or "classic" proof, but also to *explain* different aspects of the mathematical matter under consideration. The third aspect I wish to point out is that by enriching and reconfiguring a technology, one can change a hitherto "forbidden" technique into an acceptable one. More generally, the *logos* block $\Lambda = [\theta / \Theta]$, which is normally conceived as matching a determined *praxis* block $\Pi = [T / \tau]$, determines what the *praxis* block can be—a manifestation of the dialectic between *praxis* and *logos*.

A person's or, for that matter, an institutional position's praxeological equipment should not be idealized. As we have seen, the *logos* block $\Lambda = [\theta / \Theta]$ can be a serious hindrance as far as the *praxis* part is concerned. More generally, a *logos* block has always gaps and flaws, even in mathematics. To give a quick example, let us consider the familiar definition of a circle ("Circle," n.d.): "It is the set of all points in a plane that are at a given distance from a given point, the centre […]. The distance between any of the points and the centre is called the radius." You certainly speak of *the* center and *the* radius of a circle given "by center and radius." But have you ever run into a proof of the following statement, that we almost always unreflectively—but rightly—take to be true: a circle defined by a center O and a radius r has one and only one center and one and only one radius? Our praxeologies, mathematical or otherwise, are replete with self-evident "truths" and we are inclined to take the world around us for granted. It is this ubiquitous taken-for-grantedness that science and education unrelentingly question.

5. DIDACTICS AND THE DIDACTIC

According to the anthropological theory of the didactic, the aim of *didactics* as a science is to elucidate the mechanisms by which, in a given society *Š*, knowledge is diffused within institutions *I* and among persons *x*. Having arrived at this point in our analysis, we can say that the diffusion of knowledge amounts to the social diffusion of *relations to objects*, which, in turn, consists essentially in the diffusion of *praxeologies*. Before we go any further, however, we have to build upon the notions of object and praxeology—which are objects as well. Any object is in fact the product of purposeful human action, so that any object has reasons for being there. I introduce now the all-encompassing notion of "a *work*" (generically denoted by the letter *w*), which can be regarded as a generalization ("Work," 2009-2016) of the more common notion of a "work of art"—also called, according to *Wikipedia* ("Work of art," n.d.), "artwork, art piece, piece of art or art object." The notion of a work *w* applies to every detail of a praxeology as well as to larger constructs, such as a whole discipline $\mathcal{D}$. A praxeology $\wp = [T / \tau / \theta / \Theta]$ is called a *pinpoint* praxeology, because it focuses on a "point," to wit the particular type of tasks *T*—one may say that $\wp$ is "built around" *T*. I have already alluded to the *ecology* of praxeologies, i.e., the bunch of conditions of all kinds and origins that bear upon a given praxeology or set of praxeologies. In a society $\check{S}$ and, more specifically, in an institution *I*, people make decisions, do things to manage all kinds of objects—of works—, including praxeologies. The aim of this managing activity is to modify conditions under which this or that work "lives"—in other words, this aim is to change, however little it be, the ecology of the work in question. This "handling" of the ecology of works is the *economy* of works. (According to Ayto, the Greek suffix *nomos*, that means "rule, law, custom, usage," comes from the verb *némein* "manage.") As we shall see, the ecology and economy of works, which includes the ecology and economy of the didactic relating to them, constitute the horizons of ATD.

An essential component of our framework is the ecology and economy of *praxeologies*. In a given society $\check{S}$, a welter of activity goes on about praxeologies, whose major effect is the ubiquitous "amalgamation" process that joins together a number of praxeologies. The historically contingent institutional amalgamation of praxeologies is first achieved by aggregating pinpoint praxeologies into so-called *local* ones sharing the same *logos* block $[\theta / \Theta]$, common to a number of *praxis* blocks $\Pi_i = [T_i / \tau_i]$, so that a local praxeology can be written thus: $\wp' = \sum_i [T_i / \tau_i / \theta / \Theta]$. Local praxeologies, also called local praxeological *organizations*, can themselves be amalgamated into so-called *regional* praxeologies (or regional praxeological organizations), aggregating local praxeologies that share a common *theory* Θ: $\wp'' = \sum_{j,i} [T_{ji} / \tau_{ji} / \theta_j / \Theta]$. In turn, regional praxeologies can be amalgamated into *global* praxeologies, written as follows:

$$\wp''' = \sum_{kji} [T_{kji} / \tau_{kji} / \theta_{kj} / \Theta_k].$$

As we are going to see now, however contingent they are, global praxeologies are the stuff that disciplines are made of.

Roughly speaking, a pinpoint praxeology $\wp$ corresponds, in a given mathematics curriculum, to what we shall call a *subject* of study, often labeled by a theorem—i.e., a technological element—thought characteristic of this subject of study, such as the "Pythagorean theorem" when the "subject" in question boils down to the type of tasks consisting in calculating the length of a side of a right triangle given the length of the two other sides. A *local* (praxeological) organization can be identified with the praxeological elaboration of a *theme* of study, for example that of calculating distances in space. A *regional* organization will be likewise identified with a *sector* of study, for example that of "metric" geometry—as distinct from "Cartesian geometry," which can be considered another sector of study. A *global* organization corresponds to a *domain* of study, for example "geometry." We should add to this a fifth level, that of the discipline $\mathcal{D}$, in this case mathematics, which encompasses different domains (here for example geometry, algebra, trigonometry, etc.) The gentle reader should not be disconcerted by the fact that this five-tier model of (mathematical) knowledge may appear somewhat arbitrary. It is not so much the modeling as the reality modeled—which is the product of human activity—that exhibits (reasoned) arbitrariness. In fact, the five-tier model of taught knowledge—subject, theme, sector, domain, discipline—seems to be relevant to most disciplines, even when their terminology differs from ours—we all know for instance that the word "subject" may be applied to a whole discipline, while words not used here, such as "area (of study)" or "field (of study)," are available in that vein as well. Likewise, the different mathematical entities mentioned above may differ in time and space according to a changing curriculum—a point I will not dwell on here. More generally, it should be stressed that amalgamation processes are in no way "natural": they are the product of human work. Like other theoretical approaches in the social sciences, ATD endeavors to "deconstruct" the illusion of naturalness of human works, which is maybe the central illusion that we have to come to grips with in the scientific approach to knowledge and its diffusion.

In ATD, one studies the conditions that favor or preclude the diffusion of knowledge. A peculiar feature of this approach is that ATD does not claim to explain the diffusion or nondiffusion (or withholding) of praxeologies by referring to a confined subset of conditions—for example by limiting the scope of didactic analysis relating to some didactic system $S(X, Y, w)$ to characteristics of the students $x \in X$, the "didactic helpers" (teachers, tutors, etc.) $y \in Y$, and, last but not least, the didactic stake w (where w is *any* work), even if these three factors remain fundamental. A great number of didactic analyses lead to consider a whole *scale of levels of didactic co-determinacy*. One such level, which is the lowest one, is naturally the level of *didactic systems*. Immediately above are two levels usually taken into account in pedagogic analyses—the level of *pedagogy* and the level of *school*, usually represented by the following diagram:

…

⥯

School

⥯

Pedagogy

⥯

Didactic system

Figure 4. The scale of didactic co-determinacy (1)

The tag of each level ("Didactic system," "Pedagogy," etc.) is in the singular. But, of course, there exist lots of didactic systems, many pedagogies, and a wide diversity of schools. Each level is the seat of conditions specific to that level. However these conditions are, up to a point, effective in the entire social space: a condition grown out of society's specificities, for example, may—in a variable extent—bear upon realities of *any* other level.

The broadening of the scope of analysis thus brought about by ATD is yet more visible when one looks at the higher levels of the scale, as shown below:

Humanity

⇵

Civilization

⇵

Society

⇵

…

Figure 5. The scale of didactic co-determinacy (2)

Here, the label of the highest level—*humanity* (or *humankind*)—is rightly in the singular, because there is only *one* humanity since Neanderthals vanished some 30 000 years ago. This level is definitely of paramount importance to us. Thus, numerous conditions supposed to be realized in the "subjects" of experimental psychology are most probably shared by all *Homo sapiens*—well beyond the formerly so-called WASP ("White, Anglo-Saxon, Protestant") population, for example. Nevertheless, some conditions are (or have been) peculiar to some past or present *civilization*. Other conditions are realized only in a given *society*, while they are absent from other societies belonging to the same civilization. More generally, in going down the scale of didactic co-determinacy, we meet with conditions of greater specificity, until the final "rung," which is the most specific of all, that of didactic systems.

At all levels, among the conditions of interest to didactics research, a great number are created *intentionally* by persons and institutions. In what follows, instead of repeating over and over "a person or an institution," I shall say, for short, "an *instance*," to designate indifferently either a person or an institution. Thus, when an instance u does something—some "didactic gesture," denoted by the Greek small letter δ—with the intention of helping an instance v to study—in order to know it better—some work w (which, let me repeat, can be *any* object o), we observe what we call a *didactic fact*. A didactic fact[20] ∂ is formally defined as a quadruplet (u, δ, w, v) satisfying the aforesaid quaternary relation, that we denote by $\vartheta(u, \delta, w, v)$. We then define *the didactic* in $\check{S}$ to be the (fuzzy) set[21] ⦰ of didactic facts occurring in $\check{S}$: ⦰ $= \{(u, \delta, w, v) \mid \vartheta(u, \delta, w, v)\}$. More generally, given a set of institutions $\mathcal{I}$, we shall define the didactic in $\mathcal{I}$ as the set of all the didactic facts that "take place" somewhere in $\mathcal{I}$.

The central aim of didactics is to identify and analyze the didactic in $\check{S}$ as well as in any complex of institutions $\mathcal{I}$, and to elucidate their effectiveness. The expression $\partial = (u, \delta, w, v)$ leads to raise some fundamental questions. A trivial but crucial remark in this respect is that, as a rule, the set ⦰ is *not* a Cartesian product of the form $U \times \Delta \times \mathcal{W} \times V$: for example, given u, w, and v, the "values" of δ such that $\vartheta(u, \delta, w, v)$ do depend on the triplet (u, w, v). More generally, the central question of didactics is therefore: Given the values of three of the four variables u, δ, w, and v, what can be the values of the fourth variable? This schema of questions can, in truth, generate most questions of interest in didactics. In what follows, I will briefly look into some of them.

20 Here, the symbols ∂ and ϑ can be read as "curly d" and "curly theta," respectively.

21 About the symbol ⦰, which designates the (fictitious) "*reversed* empty set," one can peruse the deletion discussion on the *Wiktionary* talk page ("Talk: ⦰," 2015).

6. Mathematics education past and present

Let $\mathcal{I}$ be a complex of institutions in a society $\check{S}$ and let Σ be a school system in $\mathcal{I}$. Let us first consider the possible "values" of v such that $\vartheta(u, \delta, w, v)$. In most societies $\check{S}$, whole segments $\tilde{v}$ of the population can be ignored by Σ and more generally by most school systems in $\mathcal{I}$: whatever the values of u, δ, and w, $\tilde{v}$ is left out of the set of admissible values of v. In particular, in traditional, ruthlessly unipolar societies, fully-developed school systems receive elite instances v and reject dominated instances $\tilde{v}$, the more so as $\check{S}$ or $\mathcal{I}$ agree to a single set of values and beliefs, shared by a social elite and imposed upon the others—especially the lower classes and the female population. Just to give an idea, let me note that, in France, during most of the nineteenth century, secondary education as we understand it today was closed to girls; and that, around 1930, the proportion of (male) students attending French high schools was still strictly less than 5%.

There was more to it than that. Even in the case of the cultural and socio-economic elites, there existed a long tradition of imposing the "values" of the variable w. For most secondary students, the knowledge imparted to them was not the knowledge they were supposed to need, but the knowledge selected according to criteria that hardly concerned them. For example, in 1800, two prominent French mathematicians, Gaspard Monge (1746-1818) and Sylvestre-François Lacroix (1765-1843) decided that calculus would *not* be taught at the secondary level, but should be reserved for the students of the "École polytechnique" created in 1794 to train engineers at the highest possible level. This explains why, in secondary math classes, during the greatest part of the nineteenth century, problems of maxima and minima were solved algebraically using what we have called Cauchy's theorem—without resorting to calculus.

This is one aspect, not at all anecdotal, of what I call the *primordial problematic* in didactics—the problem of choosing the knowledge to be taught in a given school system Σ. As hinted above, the problem of the variable w cannot be approached independently of the problem relating to the variable δ of didactic gestures—and conversely. In order to exemplify the intrinsic link between δ and w, I shall start from the primordial problematic as applied to mathematical knowledge. Let a school system Σ and an instance v, considered for occupying the student's position s in Σ, be given. What will be the (mathematical) works w such that, in Σ, we have $\vartheta(u, \delta, w, v)$, for appropriate values of δ and u? In other words, what will v be "taught" in Σ? In France at least—but I suspect the same kind of mathematics curriculum gained currency at about the same time in many countries—the mathematical knowledge taught at the secondary level was the mathematics supposedly needed to become in a near future a graduate of the École polytechnique and then to work as an engineer and, maybe, as a professional mathematician—which was notably the case of Cauchy and, some seventy years later, of Poincaré. I have proposed a simple—even simplistic—model that helps to understand the major issue I want to bring to the fore. Let P be the country's general population, P_1 the tiny (but powerful) subpopulation of (productive) mathematicians, and P_2 the larger subpopulation of all those who have studied mathematics at tertiary level or have at least been initiated to what is sometimes called "Further Mathematics" (n.d.)—engineers, physicists, secondary school mathematics teachers, college and university teachers of physics and teachers of chemistry, econometricians, etc. The problem of training students aiming at becoming engineers or mathematicians can be regarded as pragmatically solved since at least the end of the nineteenth century. Of course, a working solution can always be sensibly improved—nothing is ever good enough (Hull, 1953). But the real problem lies elsewhere—and it is an undying problem.

Let us consider the subpopulation P_3 defined by $P_3 = P \setminus (P_1 \cup P_2)$, so that we have $P = P_1 \cup P_2 \cup P_3$. Roughly speaking, the members of $P_1 \cup P_2$ are the "math people" and the members of P_3 are the "nonmathematicians" or "non-math people." The problem that seems to be never-ending is precisely that of the *mathematical education of non-math people*—a problem that, in general, math people find uninteresting and which leaves them eloquently silent. It is not only a theoretical and practical problem: it is *a fundamental problem of our time*.

I shall posit here that every person x and every institutional position p needs *some* mathematical knowledge—in other words, that their praxeological equipment $E(x)$ and $E_I(p)$ should contain mathematical praxeologies relevant to their activity. This problem can be fruitfully generalized. For any discipline $\mathcal{D}$, there are $\mathcal{D}$-people and there are non-$\mathcal{D}$-people and we can write: $P = P_1^{\mathcal{D}} \cup P_2^{\mathcal{D}} \cup P_3^{\mathcal{D}}$ (we leave the reader identify on one's own the three subpopulations $P_i^{\mathcal{D}}$). The problem we have to cope with is the problem of the $\mathcal{D}$-education of non-$\mathcal{D}$-ist students—i.e., students who do not intend to specialize in the discipline $\mathcal{D}$—, whose praxeological equipments should, however, contain some relevant $\mathcal{D}$-praxeologies. More generally, for the better life of the persons and institutions concerned, any personal or positional praxeological equipment should contain "mixed" praxeologies, made up of constituents taken from different disciplines $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$.

How was that problem "solved" or rather *masked*—in fact, the problem remains open to this day—in the case of mathematics education at the secondary level? So far non-math students have received essentially the same mathematics education than future math-people, their education being organized around the needs of future members of $P_1 \cup P_2$, with contents qualitatively equivalent, although differing more or less in quantity and "depth." Of course, non-math students have always been "free" to succeed or to fail! Historically, the "primordial" problem—what (mathematical) knowledge do they really need, and for what?—was circumvented thanks to an expedient "doctrine" resting upon the following remark. Any piece of knowledge $\mathcal{k}$, any work w may be considered from two different angles. In the first place, $\mathcal{k}$ can be sought after for what I call its *inherent formative utility*. One "normally" studies quadratic equations or English, for instance, to *solve* quadratic equations and to *speak* English, respectively. In such a case, the expected formative effect is specific to the work studied: the utility of studying a work w is therefore said to be *inherent in w*—it inheres within w, hence the name "inherent formative utility." Inherent formative utility becomes a problem when v is taught knowledge that has little or no utility to manage v's life responsibly—which was the case with the traditional, scientifically elitist, secondary curriculum. This conflict situation was "solved" using two "bourgeois" dogmas. The first was the battle cry "knowledge for knowledge's sake," which was advocated by a large middle-class and aristocratic majority and opposed unconvincingly by a wee minority of free spirits—for example, in his (incomplete) book *Philosophy in the Tragic Age of the Greeks* (1962, pp. 30-31), Friedrich Nietzsche (1844-1900) states that "an unrestrained thirst for knowledge for its own sake barbarizes men just as much as a hatred of knowledge." The second was the notion I have called the *transcendent* formative utility of a piece of knowledge $\mathcal{k}$, supposed to be independent of its *inherent* formative utility, even though some pieces of knowledge are thought to have a greater transcendent formative utility. In this case, one does not study quadratic equations or Latin to solve quadratic equations or to read Latin. Their study is held to be justified by the much trumpeted belief that it is in and of itself highly "formative"—so that, by the way, the knowledge studied can be readily forgotten once the formative virtue that its study contains has operated.

This "solution" to the primordial problem leaves the problem entirely open. In what follows, I will deliberately leave aside the "transcendentist" point of view. The question "What works w should v study?" is in fact closely linked to the question of the variable δ, i.e., to the didactic gestures that are germane to the pair (v, w)[22]. I would like first to mention a few well-known types of didactic gestures. "To give a lesson," "to give a lecture," "to explain" are types of didactic gestures on the part of a teacher y, while "to do one's homework," "to look for an explanation of," "to ask the teacher about" are didactic gestures on the part of a student x. These types of "gestures" are in fact types of tasks or parts of them. "To answer a student's question" or "to ignore a student's question" are other types of didactic gestures. We can count as such the fact for a teacher "to give a big wink by way of answer to a student's question," or "to give students an URL," etc. As one can easily imagine, this list could be continued almost indefinitely.

Given v, a solution to the primordial problem must certainly rest on a watchful exploration of v's potential needs and wants, from which the works for study w will emerge. Beyond that, given v and w, we shall have to determine the most appropriate types of didactic gestures δ. One crucial remark must be made here. In the "transcendentist" problematic, a work to be studied is less important that the study that is made of it. To study w means generically that, under the supervision and with the help of a teacher y, a student x considers a number of questions Q about w and manages to arrive at a reasoned answer A—a pair (Q, A) being called an *item of knowledge*. If the school $\sigma \in \Sigma$ regards A as an admissible answer to Q, if not as *the* answer to it, then it will be said that *learning* has taken place. In this didactic paradigm, learning is essentially the criterion that allows teachers to satisfy themselves that the students have indeed studied w, which is the reason why the knowledge learned can, after a while, be completely and serenely forgotten.

It is a very different story when one passes from the transcendentist point of view to the "inherentist" stance. In this latter case, the knowledge learned about some work *w does* count and should *not* be forgotten: you don't study w just to have done it, but, it is hoped, to know w permanently. While in the transcendentist approach to works you don't have to use w, to handle it or draw upon it, in the inherentist approach you will *use* it, so that you have to know what it is for. In the former case you can safely ignore the *raisons d'être*, the reasons for being, of w. What are angles for? And parallelograms? And fractions? You simply have to do the things you are ordered with them—calculate this angle, prove that this quadrilateral is a parallelogram, etc. By contrast, in an inherentist approach to knowledge, you have to know *why*, for instance, it is interesting to know the value of an angle and how you can use the fact that this quadrilateral is a parallelogram, etc.

The long historical dominance in most school systems of the transcendentist relation to knowledge has imposed to this day what I have called the *paradigm of visiting works*. A (mathematical) work is "visited" by a class under the supervision of the teacher as if it were a monument, even a masterpiece, that, however impudently, we are expected to revere and bow to. This leads to what I have called the "monumentalization" of the curriculum. Now when, to the contrary, we adopt the inherentist stance, things change almost completely.

The first historical step in this direction was taken a number of decades ago when the French "modern" didacticians, following in the wake of Guy Brousseau's pioneering work (1997), set to tackle the general

22 To keep this paper within reasonable limits, we knowingly leave out the outstanding issue of the variable u.

basic problem of didactics: Given a work w, find a question Q the study of which will, if not generate, at least leads one to come across w, regarded as a key resource to arrive at an answer A to Q. Such was the first systematic and effective effort to "demonumentalize" the mathematics curriculum. Acting on the basis of Brousseau's theory of didactic situations, another step was then taken by ATD, in which the notion of "study and research activity" (SRA) was developed. This notion was soon reworked to give birth to the notion of "study and research *path*" (SRP) in which the question Q, which at the start seemed to be a mere foil to the work w, soon assumed greater significance.

7. INQUIRING ABOUT THE WORLD AROUND US

Today, in the research that some of us are pursuing, the question Q is the alpha and omega, while the works w to be studied in the course of inquiring about Q are no longer "given" by an omniscient instance but are determined by the very logic of the inquiry about Q. In this perspective on "knowing by inquiring," it was soon observed that a question Q itself is also a product of the human mind—a very precious one indeed—and is therefore a *work* in its own right. The work and toil done along these lines have been encapsulated in the notion of the *paradigm of questioning the world* and the so-called "semi-developed" *Herbartian schema*, that we write as follows: $[S(X, Y, Q) \curvearrowright M] \hookrightarrow A^{\heartsuit}$. We denote by Q the *question* studied by the "class" $[X, Y]$ and by $A^{\heartsuit}$ the *answer* that the class's inquiry struggles to arrive at. The heart in superscript in $A^{\heartsuit}$ means both that $A^{\heartsuit}$ is dear to the class's "heart" in that it is the most optimal answer to Q that they could consider, *and* also that—consequently—it is this answer that will be "at the heart" of the class's activity in the times to come—$A^{\heartsuit}$ will be the "official" answer to Q in the class $[X, Y]$, until a resumption of the inquiry on Q someday leads the class to adopt a new, "updated" answer $A^{\heartsuit\prime}$.

This schema can be further "developed" by replacing the letter M with the following expression, which we shall briefly dwell on:

$$M = \{A_1^{\lozenge}, A_2^{\lozenge}, \ldots, A_m^{\lozenge}, W_{m+1}, W_{m+2}, \ldots, W_n, Q_{n+1}, Q_{n+2}, \ldots, Q_p, D_{p+1}, D_{p+2}, \ldots, D_q\}.$$

First of all, the Herbartian schema—in both its semi-developed and fully developed forms—should make it plain that, in order to build up an answer $A^{\heartsuit}$ to Q, the didactic system $S(X, Y, Q)$ has to concomitantly bring into existence a *didactic milieu M* made up of all the "tools" the use of which seems indispensable or at least useful or simply enjoyable to the class $[X, Y]$. By contrast, the answers denoted by $A_i^{\lozenge}$ are "ready-made" answers to Q that the team of investigators X, supervised by Y, have discovered in the institutions around them: they are (other) *institutional* answers to Q. The lozenge ◊ in superscript reminds us that the answer $A^{\lozenge}$ is, so to speak, "hallmarked" by an institution or by one of its positions, of which it is currently the "official" answer to Q. The W_j are *works* specifically drawn upon to make sense of the $A_i^{\lozenge}$, analyze and "deconstruct" them, bring appropriate answers to the questions Q_k, and, last but not least, build up $A^{\heartsuit}$. The Q_k are the questions induced by the study of Q, the $A_i^{\lozenge}$, and the W_j as well as the questions raised by the construction of $A^{\heartsuit}$. Finally, the D_l are sets of data of all natures—both quantitative and/or qualitative in nature, for example—gathered in the course of the system's inquiry on Q, on which the didactic system's answers to the questions under consideration partially rest.

As we shall soon see, the Herbartian schema plays a signal part as a basic technological tool to design or refashion *didactic organizations* in terms of *inquiry*. It must nevertheless be stressed unambiguously that the Herbartian schema is not only a major tool in the design and monitoring of novel didactic organizations but also a key instrument in modeling and analyzing old as well as unprecedented didactic organizations. Although the word "inquiry" may sound here as a watchword, almost as a battle cry, in the framework of ATD the notion of inquiry is the fittest tool to make sense of already existing didactic forms.

Let me indulge in an easy but crucial example, that of the traditional "lecture-based" teaching. From the point of view of ATD, "to give a lecture" is to perform a didactic gesture δ of a definite type. This type of gestures is always part of a broader didactic technique τ_∂ in which it is supposed that the students $x \in X$ and the teacher y will cooperate. More generally, in the case of a type of tasks T in which different categories of persons $\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_p$ cooperate, i.e., in a *cooperative* type of tasks T, any technique τ_T has to define the *role* of each category $\mathcal{X}_i$ as well as its *topos* (for $i = 1$ through p). The *role* of $\mathcal{X}_i$ describes what members of $\mathcal{X}_i$ have to do when taking part in the carrying out of τ_T—it specifies "the part they play" in this process[23]. By contrast, if performing τ_T on a given task $t \in T$ consists in performing the conjunction of tasks $t_1 \wedge t_2 \wedge \ldots \wedge t_n$, then the *topos* of $\mathcal{X}_i$ is "the place"—which is the meaning of the Greek word *topos*, at the root of *topology*, *topography*, or *isotopic* (literally "having the same place")—made up of the types of tasks $T_1, T_2, \ldots, T_n$ on which members of $\mathcal{X}_i$ are supposed to act "on their own," so to speak "uncooperatively," and (therefore) in full responsibility of the tasks $t_j \in T_j$ that it behooves them to carry out. Clearly, with respect to a given type of tasks T, the role of a given category of actors *includes* their *topos*, which is usually a small part of their role. This generic model applies in particular to the case of a didactic system $S(X, Y, Q)$. In such a case, we have essentially *two*[24] categories of actors, the category of students, $\mathcal{X}$, and the category of "teachers," $\mathcal{Y}$.

Two remarks are needed here. No didactic gesture δ can be didactically analyzed—and still less didactically "criticized"—as an independent didactic entity. In every case, we have to approach it as an element embedded in a whole didactic technique τ_∂. In particular a *lecture* is part of a didactically cooperative praxeological organization. Many types of organizations could be taken into consideration here. The crudest of them boils down to this: the teacher lectures to the students, who try to note down the material of the teacher's presentation, with a view to studying it on their own soon thereafter. But this is *not* the only possibility: a lecture may well be a (small) part of an inquiry process in the sense we will explain a little further on. In any case, the lecture itself can be construed as follows: the teacher y has a) *inquired* about a question Q or a set $\mathcal{Q}$ of such questions, and b) established a more or less thorough *account* of this inquiry and its findings. When lecturing to students, the teacher presents to them a *report* on the inquiry thus carried out and expounds an answer to the question $Q \in \mathcal{Q}$ that we shall denote by $A_y^\heartsuit$. Here, the superscript ♥ means that this answer is the teacher's "preferred"—in the sense of being optimal with respect to a certain set of constraints—answer to Q. As is obvious, in such a didactic organization, the students are *not* expected to take part in the inquiry proper: they receive a ready-made answer $A_y^\heartsuit$ and have to accept, understand and adopt it as *the class's answer*. In this case, the Herbartian schema can be rewritten thus: $[S(X, Y, Q) \curvearrowright M] \hookrightarrow A_y^\heartsuit$, while the didactic milieu M is made up of elements either already available in $[X, Y]$ or especially

23 According to the *Online Etymology Dictionary* ("Role," 2001-2016), the word *role* comes from the French word *rôle*, which originally applied to the "roll (of paper) on which an actor's part is written."

24 In some cases, it may be justified to distinguish subcategories of $\mathcal{Y}$ (such as "teachers," "tutors," etc.).

brought about by the teacher's lecture. But there is more to it than that.

In the kind of didactic organizations that we just described, one fact is self-evident: what is called in ATD the *topogenesis* of knowledge—i.e., of the answers $A^{\heartsuit}$ to the question Q studied—is severely tilted towards the teacher's *topos*, while the students have only to acknowledge without much ado the possibly idiosyncratic answer $A^{\heartsuit}_y$ propounded by the teacher. This structural imbalance generates, in the long run, didactic oddities of which the most anomalous lies in the fact that *questions and consequently answers vanish*. A lecture then ceases to be an outright study of *questions*: it addresses "topics," deals with "subjects," and but rarely struggles to explicitly answer explicit questions. Usually, however, lectures virtually contain answers to questions; but these questions remain implicit and go unnoticed, so that the "answers" implicitly brought forward by the lecture are no longer answers—they are "utterances" that *could* be understood to be snippets from potential answers to elusive questions. We thus arrive at a type of didactic organizations still subsumable under the Herbartian schema but that we are entitled to label as *degenerate* didactic organizations.

By contrast, I would like to conjure up a quite different type of didactic organizations, which I will call, tentatively[25], *knowing-through-inquiry* didactic organizations. Let us take an easy mathematical example, the generating question Q of which is: "What is the 200th decimal of the number $\frac{1}{371}$?" (I have repeatedly used this type of questions as a starter with eighth-graders in a workshop on "Inquiring on the Internet.") We thus suppose the existence of a didactic system $\boldsymbol{S} = S(X, Y, Q)$. The initial question Q must, in fact, be understood to mean: "What does the examination of the persons and institutions that express themselves on the Internet allows us to say—without other guarantee—about the 200th decimal of the number $\frac{1}{371}$?" It is this question $\tilde{Q}$ that will be the real focus of $\boldsymbol{S}$'s inquiry. One can first think of using an online calculator to get a decimal expansion of $\frac{1}{371}$. The question Q_k that ensues is therefore: "Which online calculators are best suited to $\boldsymbol{S}$'s inquiry?" The "fuzzy" set of calculators used will be a significant part W_j of $\boldsymbol{S}$'s didactic milieu M. The most immediately accessible online calculator, I mean the built-in calculator provided by the Google search engine, gives 0.00269541778, which hardly allows $\boldsymbol{S}$ to come to a conclusion. What is needed is a calculator that reports a really *large* number of digits. Because the last digit is unsure (due to a possible rounding effect), $\boldsymbol{S}$ has to look for a calculator that displays at least 201 decimal places in its edit box. One such online calculator[26] is the "Big Online Calculator," as we can see on the following capture (fig. 6). The "Big Online Calculator" will be a part of $\boldsymbol{S}$'s answer $A^{\heartsuit}_k$ to the question Q_k referred to earlier and one of the works W_j constituting $\boldsymbol{S}$'s didactic milieu M.

25 I avoid on purpose the flourishing use of compound adjectives like *inquiry-based*, *inquiry-centered*, *inquiry-focussed*, etc.
26 It can be found at http://www.ttmath.org/online_calculator.

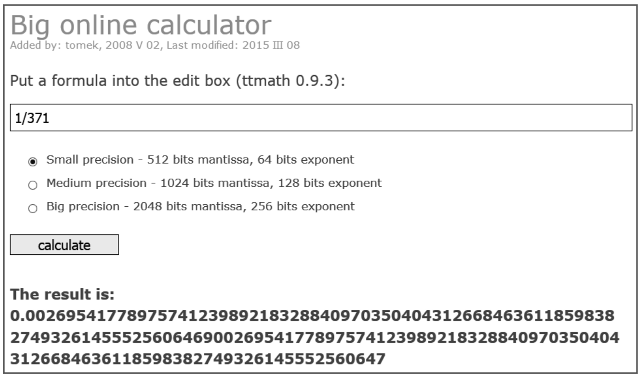

Figure 6. Using the “Big Online Calculator”

To make the most of it, $\boldsymbol{S}$ has to be able to determine the 200th digit in the list displayed, which is another question Q_k. The answer to this question is a technique τ_0 (or rather a praxeology $\wp_0$, i.e., another work W_j) that can be described as indicated hereafter. One can copy the digits displayed and paste them into a Word file, to arrive at the following sequence of digits:

00269541778975741239892183288409703504043126684636118598382749326145552560646900
269541778975741239892183288409703504043126684636118598382749326145552560647

The Word Statistics tool lets one know the number of digits displayed. In this case, there are 155 digits, i.e., less than the 201 needed. It is therefore necessary to use the calculator’s “medium precision” function, which gives the following sequence of digits:

00269541778975741239892183288409703504043126684636118598382749326145552560646900
26954177897574123989218328840970350404312668463611859838274932614555256064690026
95417789757412398921832884097035040431266846361185983827493261455525606469002695
417789757412398921832884097035040431266846361185983827493261455525606

The number of digits reported here is 309. By trial and error, using the Word statistics tool, one can select the first 200 digits:

00269541778975741239892183288409703504043126684636118598382749326145552560646900
26954177897574123989218328840970350404312668463611859838274932614555256064690026
9541778975741239892183288409703504043126

The conclusion follows: according to the “Big Online Calculator,” the 200th decimal of $\frac{1}{371}$ is a 6. In order to be surer of this result, $\boldsymbol{S}$ can resort to other online calculators. The so-called “Big Number Calculator” allows its users to choose the number of decimals displayed[27]. We thus arrive at the outcome shown below (Figure 7), which confirms the result obtained.

27 The “Big Number Calculator” can be found at https://thenerdshow.com/calculator.html.

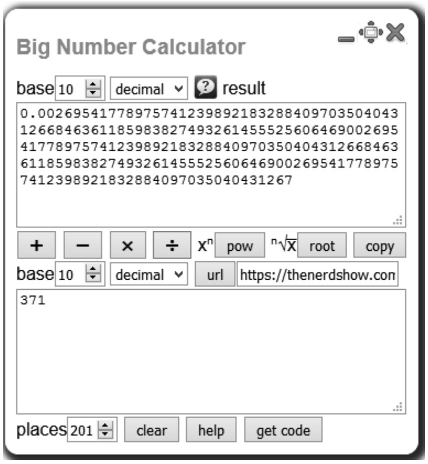

Figure 7. Using the "Big Number Calculator"

S can resort to yet other calculators, for example the "Big Integer Calculator," which, as its name suggests[28], gives the *integer* quotient of two integers x and y. If one takes $x = 1$ and $y = 371$, this calculator gives back $x \backslash y = 0$. If, however, one sets $x = 10^{202}$ and $y = 371$, the outcome is as shown below (Figure 8).

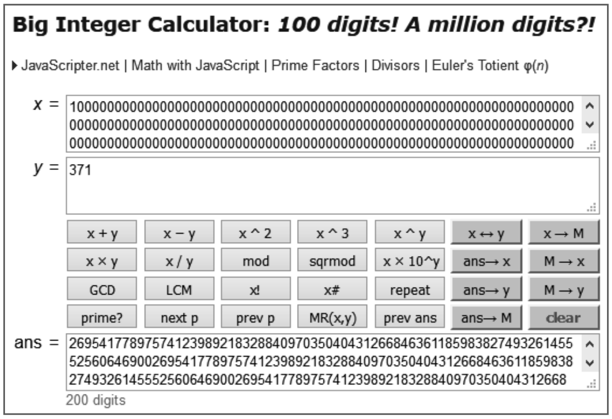

Figure 8. Using the "Big Integer Calculator"

The confirmation is confirmed!

This result can also be validated by empirical observation, which suggests that the decimal expansion of $\frac{1}{371}$ is *periodic* with the following "repetend":

002695417789757412398921832884097035040431266846361185983827493261455525606469

The length of this repetend, i.e., the period of the decimal expansion, is equal to 78. By Euclidean division, we have $200 = 78 \times 2 + 44$. Consequently, the 200th digit in the decimal expansion of $\frac{1}{371}$ is the

28 This calculator can be found at http://www.javascripter.net/math/calculators/100digitbigintcalculator.htm.

same as its 44th digit, which we can determine employing the technique already used: the first 44 digits are

00269541778975741239892183288409703504043126

so that the 44th digit is a 6. This technique τ_1 as well allows one to determine, say, the 200000th digit. Since we have $200\,000 = 78 \times 2564 + 8$, the digit we are looking for is the 8th in the repetend, i.e., 1.

The technique τ_1 has of course a much wider scope than the technique τ_0. However, it is no longer valid when we consider an *irrational* number like π. In this case, however, τ_0 is still available: using the "Big Online Calculator," we can conclude, in this case, that the 200th decimal is a 6 (again!). But if we look for the 200000th decimal place of π, the "Big Online Calculator" becomes useless. It is nevertheless true that the Internet is rich in resources that will allow one to persevere in using the rough-and-ready technique τ_0. In this perspective, interested readers in a hurry can, for example, watch an online video called "First 200000 digits of Pi" (Tivadar, 2009). But a further key remark is in order here.

While the result arrived at concerning the fraction $\frac{1}{371}$ seems beyond doubt, we may feel ill at ease with long lists of digits of π displayed on a webpage without any comment or justification. A revealing episode in the history of the study of π occurred in the work of the British amateur mathematician William Shanks (1812-1882): Shanks is known to have calculated in 1873 the number π to 707 places, which was the longest expansion of π at the time —about a century before the arrival of the digital electronic calculator. It was not until 1946 that another British mathematician, D. E. Ferguson, announced that a mistake had crept into Shanks's calculations—it had affected the 528th and all subsequent digits (O'Connor & Robertson, 2007). In "π room" at the "Palais de la découverte" (Paris's "Discovery Palace") created in 1937, visitors could see on the wall the 707 digits of π calculated by Shanks. In 1949, soon after the detection by Ferguson of the error affecting the 528th place, the digits were corrected accordingly. The moral of the story is that *all* results in *any* inquiry must be checked carefully, and even held to be provisional until the evidence seems indisputable.

Before moving towards some concluding developments, I want to succinctly sum up the main practical aspects of a subclass of the whole class of inquiries. Firstly, let us pay honor to whom honor is due: an inquiry begins with a question Q, the *generating question* of the inquiry, to which the didactic system $\boldsymbol{S}$ formed around Q will try to provide a "solid" answer $A^{\heartsuit}$. As an illustrative case study, let us consider the following question Q: "When was the backslash \ used for the first time to denote the integer quotient? By whom (person or institution)?" A second practical aspect of an inquiry is the *keeping of an inquiry logbook* $\boldsymbol{\mathcal{L}}$. $\boldsymbol{S}$ will first note in $\boldsymbol{\mathcal{L}}$ the starting date of the inquiry—say, Tuesday, October 12th, 2016—and its generating question Q. A third decisive aspect consists in *submitting a query to some search engine*. Let us choose Google and the following query: ⟨Backslash and integer division⟩. The fourth point consists in opening a number of Web pages and taking notes from them. Here for example we first come across the following "message" (from a Web page regarded as media): the backslash is used in the programming language called Visual Basic to denote the integer division (Microsoft, 2016). Another Web page contains the following comment, which confirms that Visual Basic uses the backslash as already said (Dahlgren, 2005):

> Many Microsoft Project users are not professional programmers so they might not be aware of some of the basics of visual basic. One of them which surprised me when I first ran across it was the

> "integer division" operator. Now most people know the typical add +, subtract -, multiply *, and divide / operators and what results they bring. But there are really two more which are quite useful in certain situations.
> The first is the integer division operator which is a backslash "\". Do not confuse this with the forward slash "/" which is used for regular division. The results of this operator are that division takes place as usual except any non-integer remainder is discarded.
> Here are a couple of examples to illustrate.
> 10/4 = 2.5
> 10\4 = 2
> 5.423/1 = 5.423
> 5.423\1 = 5
> As you can probably guess, integer division is a handy way of dividing and rounding down in a single step.
> Another related operator is the MOD operator. It is similar to integer division only it returns only the remainder. Here are a couple of examples.
> 6 MOD 4 = 2
> 12 MOD 4 = 0
> By putting them together you can break numbers into their component parts.

The author of this post then adds an example of how to use these operators:

> Doing date math is an easy way to see how this works. Let's let "Days" be a number of days. We want to know how many weeks and how many days it is. The following formula would return how many weeks and how many days there are in that amount of time.
> Days\7 & " Weeks, " & Days MOD 7 & " Days"
> If Days is 23 days, then the result would be:
> 3 Weeks, 2 Days

At this point in our incipient inquiry, one may be tempted to make the conjecture that the use of the backslash to denote the integer quotient was, if not initiated, at least made popular by the Visual Basic programming language. But let us continue our investigation. In an answer to the question "What is the reason for having '//' in Python?" we learn that the programming language Python uses a "double forward slash," //, to denote integer division ("In Python," 2008). A comment by an answerer is of value to us: "I actually like this style better... I can remember in at least one language I've used (VB?) the differentiating factor was / vs \ ... but I could never remember which was which!" This being so, the Wikipedia article titled "Division (mathematics)" features the backslash as yet another notation for division (not for *integer* division):

> Division is often shown in algebra and science by placing the *dividend* over the *divisor* with a horizontal line, also called a fraction bar, between them. For example, a divided by b is written $\frac{a}{b}$. This can be read out loud as "a divided by b", "a by b" or "a over b". A way to express division all on one line is to write the *dividend* (or numerator), then a slash, then the *divisor* (or denominator),

> like this: a / b. This is the usual way to specify division in most computer programming languages since it can easily be typed as a simple sequence of ASCII characters. Some mathematical software, such as MATLAB and GNU Octave, allows the operands to be written in the reverse order by using the backslash as the division operator…

It seems that the reverse order ($b \setminus c$) derives from long division, where students have to consider the number of times that b (the divisor) "goes into" a (the dividend), which is usually written[29] $b\overline{)a}$ or $b\overline{|a}$.

The next result of Google is the Wolfram MathWorld article titled "Integer Division," which begins as follows (Weisstein, 1999-2016):

> Integer division is division in which the fractional part (remainder) is discarded is called integer division [*sic*] and is sometimes denoted \. Integer division can be defined as $a\backslash b \equiv \lfloor a/b \rfloor$, where "/" denotes normal division and $\lfloor x \rfloor$ is the floor function. For example, 10/3 = 2 + 1/3, so 10\3 = 3.

The use of the backslash for integer division is not, even here, considered routine—the integer division is "sometimes" denoted by a backslash.

We can pause to reflect on the material gathered until now. When was the Visual Basic software released? A reasonably solid answer can be found on the Internet: 1991. What about the backslash in general? According to the *Wikipedia* article titled "Backslash," the history of the introduction of the backslash could be narrated as follows ("Backslash," n.d.):

> Bob Bemer introduced the "\" character into ASCII on September 18, 1961, as the result of character frequency studies. In particular the \ was introduced so that the ALGOL boolean operators ∧ (AND) and ∨ (OR) could be composed in ASCII as "/\" and "\/" respectively. Both these operators were included in early versions of the C programming language supplied with Unix V6, Unix V7 and more currently BSD 2.11.

In a paper titled "How ASCII got its backslash," Bob Bemer (1920-2004) calls the backslash "my character." As concerns integer division, the same *Wikipedia* article simply observes that "in some dialects of the BASIC programming language, the backslash is used as an operator symbol to indicate integer division." We thus arrive at the following conjecture: "The use of the backslash to denote integer division has emerged in the wake of the BASIC programming language and its dialects, and, although sporadically observable, has not yet gained universal recognition." The BASIC programming language first appeared on May 1, 1964. Did it already include the backslash to denote integer division? I leave it to the interested reader to inquire about this question.

I shall stop here, i.e., in its beginning stages, the inquiry about the life and works of the backlash. But I would like to stress two other points concerning any inquiry whatsoever. Firstly, at every stage in an inquiry, besides making notes of all the "findings"—including conjectures—that might prove relevant to the pursuit of an answer $A^{\heartsuit}$, $\mathcal{S}$ must pause to write "interim reports" that will hopefully converge towards the answer $A^{\heartsuit}$ wished for. Secondly, an inquiry is not conducted only through systematic explorations, as the preceding developments might lead one to believe. Quick, haphazard, unsystematic, random surveys seem

---

29 In the English-speaking world.

to be an essential complement to bring to the fore potential key facts not yet identified. A Web page titled “Symbol or notation for quotient operator” begins as follows (“Symbol or notation,” 2014):

> I’m trying to describe an algorithm in pseudocode where I’ve used the integer division operator. In VB.NET, the language I’m using, the operator used is “\”, but I don’t know if this is unambiguous to the reader that this symbol means “integer division”. I’ve also seen “div” used. I can’t use TeX or anything complicated format-wise. Is there an accepted symbol or notation for integer division operator, so that it very clearly means “integer division” and not regular division?

Although this question sounds familiar to us at this point of our inquiry on the backslash, it will also remind the seasoned reader of a once well-known fact: the use, in several programming languages, of the abbreviation “div” to denote integer division, as was the case with Pascal (“Assignment,” 2010-2016), the first version of which appeared in 1970. Nothing ensures us that such a “new” fact will not prove a useful clue to the answer we are after.

8. $\mathcal{D}$ FOR NON-$\mathcal{D}$-ISTS OR THE FUTURE OF (MATHEMATICS) LEARNING AND TEACHING

In this last section I will refer mainly to mathematics. But the analyses proffered extend to any discipline $\mathcal{D}$ whatever. I have already mentioned the subpopulations $P_1^{\mathcal{D}}$, $P_2^{\mathcal{D}}$, and $P_3^{\mathcal{D}}$. These entities are, so to speak, defined in absolute terms. In particular, what we called “mathematicians” are the members of $P_1 = P_1^{\mathcal{M}}$, where $\mathcal{M}$ stands for “mathematics,” that is, they are what we have called “$\mathcal{D}$-ists,” with $\mathcal{D} = \mathcal{M}$. We now define a *relative* notion of $\mathcal{D}$-ist. In any institution I, it is often possible to observe the existence of a position that I will denote by $p_{\mathcal{D}}$, which is the position of “I-specialists in $\mathcal{D}$,” i.e., people to whom one turns in I to solve problems that supposedly pertain to $\mathcal{D}$. There will be for instance the “computer specialist” in a family, the “law specialist” in a political campaign staff, the “specialist in English” and the “specialist in statistics” in a research team in didactics, etc. Mathematics teachers are in particular the $\mathcal{M}$-ists in the schools where they teach. Up to a point, all these institutional $\mathcal{D}$-ists are interested in $\mathcal{D}$ “for $\mathcal{D}$’s sake,” while non-$\mathcal{D}$-ists, who may have to use bits or even chunks of $\mathcal{D}$, are interested in $\mathcal{D}$ for reasons foreign to $\mathcal{D}$—let us say, for $\mathcal{D}$ as a tool outside of $\mathcal{D}$. Now, as I already stated, a major problem of our time is the problem of the $\mathcal{D}$-education of non-$\mathcal{D}$-ist people. More specifically, we shall consider here the problem of the $\mathcal{D}$-education of members of ${}_3^{\mathcal{D}}$, which would at best make a member of ${}_3^{\mathcal{D}}$ into an institutional $\mathcal{D}$-ist in some “non-$\mathcal{D}$” institution I. Still more specifically, we shall focus on the case where $\mathcal{D} = \mathcal{M}$.

Before we do that, let me give a rough sketch of an overall culture that didactically agrees with the paradigm of questioning the world. Let us therefore consider the following “axiomatic” requirements on a possible society $\check{S}$:

1. $\check{S}$ advocates and supports, in all institutions I and all persons x, a permanent, dialectical interplay between the taken-for-granted and the problematizable. This means that no aspect of an institution’s or a person’s cognitive universe is beyond questioning; and, in practice, that no aspect of them will indefinitely elude being perceived by I or x as a *problem for study* in itself.
2. Every person x in one’s own right (i.e., as a subject of oneself) or as the subject of any institution I in any

position p is effectively granted the right to inquire about any question Q whatever, whenever, wherever. The same applies to institutional positions. Such inquiries can be any single person's doing or the work of a collective, be it pre-existent or constituted for the occasion, when citizens of different subjections come together to form a study community, on a permanent or occasional basis.

3. $\check{S}$ ensures that the appropriate, material and intellectual, *didactic infrastructure* is available to any person, any collective, or whatever institution, entertaining a study and research project, unless this project is judged unrealistic and unviable by an appropriate, democratically constituted committee—with a right of appeal on the part of the "applicants."
4. All citizens are encouraged to keep a personal "study and research log book" in which are recorded noteworthy study-and-research pieces in which they took part and the part they played in their achievement, in order to tentatively specify the person's "degree of exotericity" relative to a given question. A person x's degree of exotericity $\varepsilon(x, Q, Z)$ indicates "how much" x has studied the question Q during the time interval Z: roughly speaking, $\varepsilon(x, Q, Z)$ can be measured in hours of study dedicated to Q—has the person studied Q for five hours, or 50 hours, or 500 hours, or 5000 hours, for instance?
5. All citizens are deeply and earnestly convinced that, "in theory," and with sufficient time of study, they can "master" any work w as deeply as is necessary to successfully study a given question Q. In other words they can arrive at answers A_w to questions Q_w about w that are both "solid" and indisputably relevant in the study of Q.
6. Citizens are circumspect and prudent about the forbidding role sometimes attributed to $\mathcal{D}$-ists of all kind, to whom they might be pressed to relinquish their right to inquire on their own. By contrast, it is felt that *any* citizen can legitimately claim to occupy—non-professionally, though not unprofessionally—*many* positions p in *several* institutions I with quite distinct praxeological equipments $E_I(p)$.
7. In this respect, $\mathcal{D}$-ists at large, i.e., members of $P_1^{\mathcal{D}} \cup P_2^{\mathcal{D}}$, have the duty, as a component of the nation's didactic infrastructure, to help all citizens to achieve whatever study they legitimately claim to pursue.

This set of "axioms" raises the question of whether there can exist a society $\check{S}$ that satisfies such requirements. This question has two sides, related as the observe and the reverse of a single coin. There is a general, theoretical side, which boils down to the following question: Under what conditions could such a society exist? And there is a more focused, practical side, which comes down to this question: How can this or that condition be fulfilled? I have already mentioned the "primordial problem" or the identification of the praxeological needs of a given category of institutions or persons. As concerns the non-math people, I will give here only one example chosen for its simplicity. Let us consider a finite sequence of items, numbered from 1 to N, ordered according to some criterion. What is the number of items numbered from n through m (where $1 \leq n < m \leq N$)? For instance, in a series of $N = 563$ school grades on a scale from 0 to 10, sorted in ascending order, the first occurrence of the number 5 is the value with rank $n = 239$, while the last occurrence of 5 has rank $m = 302$. The number of times the number 5 occurs is given by $m - n + 1 = 302 - 239 + 1 = 64$—for one can renumber the values with rank n through m so that they are numbered from $n - (n - 1) = 1$ to $m - (n - 1) = m - n + 1$. No matter how simple, this type of tasks seems to lie beyond the boundary that circumscribes the mathematical activity of most non-math people. Another such type of tasks consists in determining the number of boxes, each of which can contain at most n eggs, needed to carry N eggs. This time, most non-math people will simply divide N by n and give $N\backslash n$ as an answer. Of course, when N is not a multiple of n, the answer should be $N\backslash n + 1$: to carry $N = 208$ in boxes of $n = 6$ eggs, you need $208\backslash 6 + 1 = 35$ boxes.

Let there be no mistake. If everyone is free to aspire to become a full-fledged mathematician, or, more generally, a complete $\mathcal{D}$-ist, I am not considering such an unlikely event here—even if one can revel in the idea of a society $\check{S}$ in which everyone would be a $\mathcal{D}$-ist, for whatever discipline $\mathcal{D}$. In the scenario I have in mind, most non-math people remain non-math people; but their relation to mathematics has changed. They no longer flee from anything that, however unassumingly, looks "mathematical." Most importantly, *with some help* from competent, idoneous institutions and persons, they can understand and even gain a good command of mathematical tools they are bound to bump into from time to time. Here is a straightforward example[30]. A history student has to read and analyze the following brief excerpt from Malthus's *An Essay on the Principle of Population* (1798)[31]:

> Taking the population of the world at any number, a thousand millions, for instance, the human species would increase in the ratio of—1, 2, 4, 8, 16, 32, 64, 128, 256, 512, &c. and subsistence as—1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. In two centuries and a quarter, the population would be to the means of subsistence as 512 to 10: in three centuries as 4096 to 13; and in two thousand years the difference would be almost incalculable, though the produce in that time would have increased to an immense extent. (p. 8)

Most history students are non-math people. Therefore they will by necessity make a selective reading of this short passage. What the student will ignore is, first and foremost, not the numbers quoted by the author, but the way they can be obtained, which, though crucial to Malthus's argument, remains implicit in the passage quoted above.

In his text, the author has, so to speak, "inscribed" meaning, which an in-depth reading should bring back to the fore. The inverse operation of "inscribing" meaning into a text consists in "exscribing" the meaning inscribed, by bringing into light the text's "innards"[32] In the case considered here, the inside of the text conceals a mathematical mechanism that an exacting reader should wish to come to grips with. It is to this end that, instead of skimming through the passage, the less mathematical reader needs some mathematical help. In a great many cases, the appropriate help can consist in "mathematical subtitles" that make explicit what was left tacit but inferable in the "manifest" text. Here is for example the skeleton of a possible "paratext" of Malthus's text:

1) The number of the population: p; the available quantity of subsistence goods: q; the available quantity of goods for each person: $r = \frac{q}{p}$.

2) At time $t = 0$, $r_0 = \frac{q_0}{p_0}$.

3) Unit of time: 25 years.

---

30 This example is taken from Sineae Kim's doctoral dissertation (2015).

31 Thomas Robert Malthus (1766-1834) first published his *Essay* in 1798; a sixth edition appeared in 1826.

32 One can regard *inscription* and *exscription* as inverse operations. The verb *to exscribe* and the noun *exscription* as used here are neologisms.

4) At time $t = 1$ (= 25 years), $p_1 = 2p_0 \wedge q_1 = q_0 + q_0 = 2q_0 \Rightarrow r_1 = \frac{q_1}{p_1} = \frac{2q_0}{2p_0} = r_0$.

5) At time $t = 2$ (= 50 years), $p_2 = 2p_1 = 2(2p_0) = 4p_0 \wedge q_2 = q_1 + q_0 = 3q_0 \Rightarrow r_2 = \frac{q_2}{p_2} = \frac{3}{4} r_0$.

6) At time $t = n$ (= 25n years), $p_n = 2^n p_0 \wedge q_n = (n + 1)q_0 \Rightarrow r_n = \frac{q_n}{p_n} = \frac{(n+1)q_0}{2^n p_0} = \frac{n+1}{2^n} r_0$.

7) If $n = 9$, i.e., after 225 years (or *two centuries and a quarter*), then $r_9 = \frac{q_9}{p_9} = \frac{10}{2^9} r_0 = \frac{10}{512} r_0$.

8) If $n = 12$, i.e., after 300 years, then $r_{12} = \frac{q_{12}}{p_{12}} = \frac{13}{2^{12}} r_0 = \frac{13}{4096} r_0$.

9) If $n = 80$, i.e., after 2000 years, then $r_{80} = \frac{q_{80}}{p_{80}} = \frac{81}{2^{80}}\ r_0 \approx 6{,}7 \cdot 10^{-23}\ r_0 \approx 0$.

One should of course flesh out the bare bones of this arithmetic by adding sensible comments—a task I confidently leave to the discretion of the concerned reader.

In the example we have just examined, the mathematics of the text shows on the surface, even if a key part of it is unacknowledged. In many cases, such a situation results from the pressure exerted in our societies towards "demathematizing" the world around us—as if mathematics were some foul matter that our brains cannot handle and our cultural innocence cannot tolerate. Allow me one more example in this regard. In a book titled *English Humour for Beginners* (1980), the Hungarian-born British author George Mikes (1912-1987) pays homage to the mathematician Charles Lutwige Dodgson (1832-1898), better known by his pen name Lewis Carroll. Among other things, Mikes writes:

> He was *one* man, a compact and complicated human unit like most of us. The logician and the writer of nonsense tales complemented each other, on most occasions beautifully and charmingly. Roger Green reports how the child actress, Isa Bowman, begged him in a letter for 'millions of hugs and kisses'. Mathematician Dodgson and artist Carroll united their forces to give this reply… (p. 104)

George Mikes refers here to Roger Lancelyn Green (1918-1987), whose book *Lewis Carroll* appeared in 1960. Mikes quotes Green's report of Carroll's reply to Isabella Bowman (1874-1958)—who later prided herself on having been "the Real Alice in Wonderland." Here is what is apparently a quotation from Green's book:

> Millions must mean 2 millions at least ... and I don't think you'll manage it more than 20 times a minute – [a sum follows]. I couldn't go on hugging and kissing more than 12 hours a day; and I wouldn't like to spend *Sundays* that way. So you see it would take 23 *weeks* of hard work. Really, my dear child, *I cannot spare the time*.

It appears that this supposed excerpt from Carroll's letter to Isabella Bowman contains the statement of a problem and the answer to that problem, but says nothing about how to arrive at this answer. Clearly, we can speculate on the cryptic remark in square brackets: "a sum follows." What is there beyond it? Here is the missing part in Mikes's quote as can be seen for example in a letter to "Isabella Bowman" in the second edition (1989) of *The Selected Letters of Lewis Carroll* edited by Morton N. Cohen with the assistance of Roger Lancelyn Green:

Christ Church, Oxford
April 14, 1890

My own Darling,

It's all very well for you and Nellie and Emsie to unite in millions of hugs and kisses, but please consider the *time* it would occupy your poor old very busy Uncle! Try hugging and kissing Emsie for a minute by the watch, and I don't think you'll manage it more than 20 times a minute. "Millions" must mean *2* millions at least.

20)2,000,000 hugs and kisses
60)100,000 minutes
12)1,666 hours
6)138 days (at twelve hours a day)
23 weeks.

I couldn't go on hugging and kissing more than 12 hours a day, and I wouldn't like to spend *Sundays* that way. So you see it would take *23 weeks* of hard work. Really, my dear Child, I *cannot spare the time*. (pp. 196-197)

Here is the deleted piece of text! Although elementary, the arithmetic in it may be not immediately clear to the average non-math reader. Let us therefore sketch some possible "subtitles." A "kiss or hug" will be denoted here by the symbol $\check{k}$. We have: $20\ \check{k}\ /\ \text{min} = (20 \times 60)\ \check{k}\ /\ \text{h} = (20 \times 60 \times 12)\ \check{k}\ /\ \text{d} = (20 \times 60 \times 12 \times 6)\ \check{k}\ /\ \text{wk}$. The "kissing and hugging" time Z is therefore given by: $Z = \dfrac{2\,000\,000\ \check{k}}{(20 \times 60 \times 12 \times 6)\ \check{k}\ /\ \text{wk}} = \dfrac{2\,000\,000}{20 \times 60 \times 12 \times 6}$ wk. Let us now indulge in the pleasure of doing *slow*, nonexpert-friendly calculations. We have: $2\,000\,000 \setminus (20 \times 60 \times 12 \times 6) = 100\,000 \setminus (60 \times 12 \times 6) = 10\,000 \setminus (6 \times 12 \times 6) = 5\,000 \setminus (6 \times 6 \times 6) = 2500 \setminus (3 \times 6 \times 6) = 1250 \setminus (3 \times 3 \times 6) = 625 \setminus (3 \times 3 \times 3) = 624 \setminus (3 \times 3 \times 3) = 208 \setminus (3 \times 3) = 207 \setminus (3 \times 3) = 69 \setminus 3 = 23$. The "hard work" that made Dodgson-Carroll recoil would therefore have lasted for a little more than 23 weeks. This is where I shall stop. One question (at least) remains open: Who were the Nellie and Emsie of whom Lewis Carroll speaks in his letter to Isa Bowman? I unabashedly urge my stoic readers to inquire by themselves on this burning issue.

Thank you all!

REFERENCES

Assignment and Operations. (2010-2016). The Lazarus and Free Pascal wiki. Retrieved from http://wiki.freepascal.org/Assignment_and_Operations

Backslash. (n.d.). In *Wikipedia*. Retrieved October 5, 2016, from https://en.wikipedia.org/wiki/Backslash

Bemer, B. (n.d.). How ASCII Got Its Backslash: Computer History Vignettes. Retrieved from http://www.bobbemer.com/BACSLASH.HTM

Bradley, R. E., & Sandifer, C. E. (2009). *Cauchy's Cours d'analyse: An annotated translation*. New York: Springer.

Retrieved October 5, 2016, from http://users.uoa.gr/~spapast/TomeasDidaktikhs/Caychy/CauchyCoursdAnalyseAnAnnotatedTranslationSources andStudiesintheHistoryofMathematicsandPhysicalSciences.pdf

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. New York: Kluwer.

Cauchy, A.-L. (1821). *Cours d'Analyse de l'École Royale Polytechnique. 1re Partie. Analyse Algébrique*. Paris: Imprimerie Royale. Retrieved October 5, 2016, from https://ia802703.us.archive.org/6/items/coursdanalysede00caucgoog/coursdanalysede00caucgoog.pdf

Circle. (n.d.). In *Wikipedia*. Retrieved October 5, 2016, from https://en.wikipedia.org/wiki/Circle

Cohen, M. N., with Green, R. L. (1989). *The selected letters of Lewis Carroll*. Basingbroke, UK: Macmillan.

Dahlgren, J. (2005, May 31). VBA - Integer Division and Mod. My project blog. Retrieved from http://zo-d.com/blog/archives/programming/vba-integer-division-and-mod.html

Extreme value theorem. (n.d.). In *Wikipedia*. Retrieved October 5, 2016, from https://en.wikipedia.org/wiki/Extreme_value_theorem

Fraction. (n.d.). *American heritage dictionary of the English Language*. Retrieved from https://ahdictionary.com/word/search.html?q=fraction

Fraction. (n.d.). *Cambridge advanced learner's dictionary*. Retrieved from http://dictionary.cambridge.org/dictionary/english/fraction?a=british

Fraction. (n.d.). *Collins English dictionary*. Retrieved from http://www.collinsdictionary.com/dictionary/english/fraction

Fraction. (n.d.). *Merriam-Webster's online dictionary* (11th Edition). Retrieved from http://www.merriam-webster.com/dictionary/fraction

Further Mathematics. In *Wikipedia*. Retrieved October 5, 2016, from https://en.wikipedia.org/wiki/Further_Mathematics

Gouvêa, F. Q. (2011, March). Was Cantor surprised? *The American Mathematical Monthly, 118*, 198-209. Retrieved from http://courses.math.tufts.edu/math61/cantor-surprise.pdf

Green, R. L. (1960). *Lewis Carroll*. London: The Bodley Head.

Holzner, S. (2010). *Physics essentials for dummies*. Hoboken, NJ: Wiley.

How to explain the difference between the fraction a / b and the ratio a : b? (2015). Mathematics Educators Stack Exchange. Retrieved from http://matheducators.stackexchange.com/questions/7281/how-to-explain-the-difference-between-the-fraction-a-b-and-the-ratio-a-b

Hull, R. (1953, February). The Mathematical training of engineers. *The American Mathematical Monthly, 62*(2), 106-108. Retrieved from http://www.maa.org/sites/default/files/pdf/CUPM/first_40years/1953-60Hull.pdf

In Python (2008). what is the difference between '/' and '//' when used for division? Stack Overflow. Retrieved from http://stackoverflow.com/questions/183853/in-python-what-is-the-difference-between-and-when-used-for-division

Kim, S. (2015). *Les besoins mathématiques des non-mathématiciens. Quel destin institutionnel et social ?* (Unpublished doctoral dissertation). Aix-Marseille University, Marseille, France.

Kirkland, C. (n.d.). *Rules of factoring*. Retrieved from http://faculty.sgc.edu/kirkland/Old Recordings/Math 99/Rules of Factoring.pdf

Malthus, T. H. (1798). *An essay on the principle of population as it affects the future improvement of society, with remarks on the speculations of Mr. Godwin, M. Condorcet, and Other Writers*. London: J. Johnson. Retrieved from http://rescuingbiomedicalresearch.org/wp-content/uploads/2015/04/Malthus-1798.pdf

Microsoft. (2016). \ Operator (Visual Basic). Retrieved from https://msdn.microsoft.com/en-us/library/0e16fywh.aspx

Mikes, G. (1980). *English humour for beginners*. London: Penguin Books.

Mitchel, O. M. (1845). *An elementary treatise on algebra: Designed to facilitate the comprehension, demonstration and application of the leading principles of that science*. Cincinnati, OH: Morgan.

Nietzsche, F. (1962). *Philosophy in the tragic age of the Greeks*. Washington, DC: Regnery Publishing. Retrieved from http://cnqzu.com/library/Philosophy/neoreaction/Friedrich Nietzsche/Friedrich_Nietzsche - Philosophy_in_the_Tragic_Age_of_the_Greeks_(tr._Marianne_Cowan_1996).pdf

Niven, I. (1981). *Maxima and minima without calculus*. The Mathematical Association of America.

O'Connor, J J, & Robertson, E. F. (2007, July). William Shanks. MacTutor Archive. Retrieved from http://www-history.mcs.st-andrews.ac.uk/Biographies/Shanks.html

Pascal, B. (1909-1914). *Minor Works: Of the Geometrical Spirit*. New York: Collier. Retrieved from https://ebooks.adelaide.edu.au/p/pascal/blaise/geom/

Poincaré, H. (1907). *The Value of Science*. New York: The Science Press. Retrieved from https://www3.nd.edu/~powers/ame.60611/poincare.pdf

Presentation delivery. (n.d.). EnglishClub. Retrieved from https://www.englishclub.com/speaking/presentations-delivery.htm

Rike, T. (2002, January 13). Maxima and minima *without* calculus. Retrieved from http://mathcircle.berkeley.edu/BMC4/Handouts/MaxMin.pdf

Role. (2001-2016). In *Online etymology dictionary*. Retrieved from http://www.etymonline.com/index.php?term=role

School. (2001-2016). In *Online etymology dictionary*. Retrieved from http://www.etymonline.com/index.php?term=school

Symbol or notation for quotient operator. (2014). Mathematics stack exchange. Retrieved from http://math.stackexchange.com/questions/711385/symbol-or-notation-for-quotient-operator

Talk: ⦸. (2015, June 3). In *Wiktionary*. Retrieved October 5, 2016, from https://en.wiktionary.org/wiki/Talk:⦸

Tivadar. (2009). First 200000 digits of Pi. Retrieved from http://thepi.blog.hu/2009/10/08/first_200000_digits_of_pi

Transpose. (2009-2016). In *Macmillan dictionary*. Retrieved from http://www.macmillandictionary.com/dictionary/british/transpose

Voltaire, F. (1901). *Candide*. New York: DuMont. Retrieved from https://ebooks.adelaide.edu.au/v/voltaire/candide/complete.html#chapter22

Weisstein, E. W. (1999-2016). Integer Division. WolframMathWorld. Retrieved from http://mathworld.wolfram.com/IntegerDivision.html

What is the reason for having '//' in Python? (2014, September 22). Stack Overflow. Retrieved from http://stackoverflow.com/questions/1535596/what-is-the-reason-for-having-in-python

Work. (2009-2016). In *Macmillan dictionary*. Retrieved from http://www.macmillandictionary.com/dictionary/british/work_2

Work of art. (n.d.). In *Wikipedia*. Retrieved October 5, 2016, from https://en.wikipedia.org/wiki/Work_of_art

Yves Chevallard
Professor emeritus at Aix-Marseille University
Marseille, France
E-mail: y.chevallard@free.fr